DOCUMENT RESUME

ED 204 120 SE 035 172

TITLE Proceedings of the 1980 Meeting [of the] Canadian Mathematics Education Study Group (Laval University, Quebec, Canada, June 6-10, 1980).
PUB DATE Mar 81
NOTE 212p.: For 1979 Proceedings, see ED 188 872. Not available in hard copy due to marginal legibility of original document.

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS Calculators; Calculus; *Educational Objectives; Educational Research; Elementary Secondary Education; Higher Education; Inservice Teacher Education; Learning Theories; Mathematical Applications; *Mathematics Curriculum; *Mathematics Education; *Mathematics Instruction
IDENTIFIERS *Canadian Mathematics Education Study Group; *Mathematics Education Research

ABSTRACT

Contained in these proceedings are: (1) two lectures; (2) reports from four working groups, three review groups, and three special groups; and (3) a list of participants. Lecture I, by Caleb Gattengo, was entitled "Reflections of 40 Years of Work on Mathematics Teaching." Lecture II, by David Hawkins, was entitled, "Understanding Understanding Mathematics." The four working groups were: (1) The Teaching of Calculus and Analysis; (2) Applications of Mathematics for High School Students; (3) Geometry in the Elementary and Junior High Curriculum; and (4) Diagnosis and Remediation of Mathematical Errors. The review groups were: (1) Calculators and the School Curriculum; (2) Mathematics Education Research: What Is It About in Canada?; and (3) Vers Une Integration de la Recherche a la Formation et au Perfectionnement des Enseignants. The special groups were: (1) Problems; (2) Uses of Gattegno's number array; and (3) Mathematical films. A total of 51 participants made this the largest meeting ever, with 18 professors from Quebec included in this number. An official constitution and an election of officers occurred at this conference. (MP)

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Canadian Mathematics Education Study Group
Groupe canadien d'étude en didactique des mathématics

Proceedings of the 1980 Meeting

Laval University, Quebec
June 6 - 10

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CMESG/GCEDM

March 1981

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CMESG/GCEDM
1980 Meeting

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* The lecture as delivered departed substantially from this text.

** The lecture is presented here by the paper "The view from below" (subsequently published in the journal "For the Learning of Mathematics"). The lecture covered the same ground, but elaborated different examples.

Summary of the Conference

This was the fourth in a sequence of annual conferences that started with the meeting sponsored by the Science Council on "The Education of Teachers of Mathematics: the Universities' Responsibility" in 1977. At the end of the 1978 meeting the group adopted its present name, at the end of 1979 meeting it appointed an acting Advisory Board and Executive Committee, and at the conclusion of the 1980 meeting it approved an official constitution and held elections of its officers. This progression indicates the growing confidence of the group in its own future.

A total of 51 participants made this the largest of the four meetings, a substantial number coming from francophone institutions in Quebec (18).

The two guest speakers, Caleb Gattegno and David Hawkins, followed the pattern set in previous meetings of joining in the other activities. Indeed, it is probably true to say that their contributions to the small groups they attended were as valuable to the conference as their lectures. Nevertheless, Gattegno baffled, intrigued and stimulated the conference with his introduction of the idea that modern technology had alerted us to the power of extremely minute amounts of energy - "nothings" - and that we could see the challenge of mathematics teaching as the task of mobilising in the learner correspondingly minute amounts of mental energy. Although the participants were unable to discuss this matter with the sensitivity and persistence that Gattegno demanded, it is likely that the idea will be heard again and will maybe prove as fruitful as its originator clearly intended.

As in previous conferences a fair stretch of time was spent in Working Groups. Feedback from participants unequivocally justifies this activity: sufficient time is given to them for the participants to feel that progress is achieved, and the groups are small enough to allow everyone the sense of getting to know the other people involved. The serious, noticeably goodhumoured and cooperative atmosphere of the conference seems to be due to the "healthy" lines of communication established in these groups. Almost no one attempts at any point to upstage anyone else - a rare phenomenon in conferences of university professors.

At least two of the Working Groups - on geometry, and on diagnosis and remediation - made good progress in their studies (the geometry group having the advantage of following fairly directly from groups in earlier conferences). The calculus/analysis group had a difficult time, mainly because it attracted people with very diverse interests in the question. A lesson learned from this occasion is that leaders of Working Groups might in future be asked to supply a brief description of the questions they would like pursued in the groups. This should help, provided it does not reduce the flexibility of the group activity.

Several one-session group meetings rounded out the programme. Although provision was left in the timetable for "ad hoc" groups, to be formed on the spur of the moment by anyone with a subject he wanted to discuss with others, this opportunity was not much taken advantage of. Perhaps the small size of the conference allows this sort of informal discussion to take place at mealtimes and in other interstices. And perhaps the members do not yet show enough initiative in shaping the conference to their own interests. Well, maybe that, too, can be learned as time goes on.

David Wheeler

CMESG/GCEDM 1980
Lecture I

Reflections on 40 Years of Work on Mathematics Teaching

Caleb Gattegno

About 1927-28, some of my cousins, about my age, and their schoolmates asked me to help them in their mathematics studies. Although I was not yet 17 I took my job seriously and so I became a mathematics teacher at the secondary school level. In 1932 I founded the Mathematics Seminar in Alexandria where a small number of people trying to educate themselves lectured to each other on selected chapters of pure and applied mathematics. That became my teaching at the graduate level, from which at least four people came out as professional mathematicians.

With my children, around 1940, when they were very young, I learned that teaching mathematics at the elementary levels was a very different activity. My failures discouraged me and for years I did not attempt to touch that population.

From 1946 to 1953 I concentrated my study of teaching and learning on the secondary and postsecondary levels. Not only did I make some progress - as can be seen from my numerous publications of those years - but I also shook off Piaget's grip on my mind. I learned to gather evidence directly from people I worked with rather than go via the work of others. My studies of consciousness started just before World War II and I was able to apply to the study of the structures of the mind the mathematical structures that had been found by mathematicians, mainly following Dedekind in 1900. The Bourbaki books were not accessible outside France at that time, but soon after 1945 I could get acquainted with them. I found that their work affected my understanding of the work of the mind but also that their ideas needed alteration if they were to serve me properly. It was then that I reached my understanding of algebra as operations upon operations. Among the Bourbaki, Dieudonné agreed wholeheartedly, in 1950, that it also described his ways of working.

By 1947, I had already propagated the notion that mathematics was the work of awareness on the dynamics of relationships per se. Soon after, algebra appeared to me as being the dynamics itself and present in all mathematical activity. Thus although mathematicians classify themselves as analysts, topologists, algebraists, probabilists, numbers theorists, logicians, and a few other subdivisions, I reached a different conclusion since I came to the meaning of mathematics through mathematisation. Some of my writings of the years around 1950 clearly hint at that.

I worked in isolation for all was new in the field and I did not seem to know how to attract the attention of my friends, those good minds I was in contact with in Great Britain and the continent of Europe. They listened with interest but left me to do the needed spade work.

In 1950 I formed the International Commission for the Study and Improvement of the Teaching of Mathematics. It counted among its original members Choquet and Dieudonné from France, Wijnsink from Holland, and myself from England. Soon after we co-opted Fletcher from England, Servais and Papy from Belgium, Puig Adam from Spain, Castelnuovo and Campedelli from Italy, and a number of others from a number of countries. The work of the Commission became known partly because of the luminaries among those named above and partly because of the two monographs I (anonymously) edited in French in 1954 and 1956 and which have been translated into Italian, Spanish, German (but not into English). As you know, the Commission still exists and meets once a year somewhere on earth.

Two offshoots of the Commission were ATAM (now ATM) in Britain, founded in 1952, and the Belgian Association of Math Teachers which also started work that year. Both involved me for a few years. In 1959 I was elected the first President of ATAM and soon after its first Honorary Member. In 1960 I resigned from the International Commission after ten years as its secretary.

I encountered teaching with films in 1949 when I met Nicolet. The following year in Debden (northeast of London) Nicolet and Jacquemard came to work for one week on teaching mathematics to adolescents. At once I liked Cantegrel's films (with Jacquemard's scenario and Motard's animation) but I soon understood that Nicolet had the better fomula. I offered Nicolet to make his work known. Unfortunately I did not manage it and when he died in 1966 although he was appreciated by a number of people it was by too few to feel he had not labored in vain.

On a lecture tour in Belgium in April 1953 I was told I had to become acquainted with the work of another obscure pioneer in the teaching of mathematis: George Cuisenaire. It proved to be a turning point in my career as a student of learning and teaching and some of you know that in a few years Cuisenaire became a household name among teachers of arithmetic all over the world. I founded eleven Cuisenaire companies, wrote twenty texts for students and teachers, met literally hundreds of thousands of teachers from 1953 to 1962, in 44 countries, on all continents.

But all this social activity is overshadowed by the discoveries I made thanks to my finding that at last I could work with young children and let them teach me what to do to reach them, understand their ways of working and from there develop the subordination of teaching to learning in a number of fields. These words came to me in 1960 when I was interviewed for the Christchurch Daily (New Zealand), and was asked to characterize my work in a few simple words. I said, "I teach what I know, and that is, to subordinate teaching to learning". At that time I had done some work in the fields of science through my "openbooks" on the Study of Energy (1957-58), in the field of reading for natives (Amharic, Hindi, Spanish in 1958, and English and French in 1959), in the field of foreign languages (1954), and, of course, in mathematics as David Wheeler, present here, can testify to.

Being busy in so many areas it was to be expected that my contributions to the teaching of mathematics would become more the expansion of what I had already done than of new findings. But life decided otherwise. In 1967 I was forced to recast the foundations of my teaching of elementary mathematics when I noticed that not all children using the rods managed to master the basic numerical facts of addition. It then occurred to me that I had to replace the rods and some manipulations of them - as I advocated for 14 years - by some awarenesses of what children can do with their fingers.

For a while I worked out the details of that approach which I published as the first chapters of my "The Common Sense of Teaching Mathematics" and in articles you may have read. Today I know for sure that that approach has made (1) an explicit use of what children do so early and so well, i.e. learn to speak (which I had not fully integrated when I asked them to work with rods), and (2) made an explicit use of two underlying mental powers which we can call the perception of equivalence and of complementarity.

This coming closer to the workings of the mind has been very fruitful, and still is. A large number of experiments with very young children and with so-called learning disabled, have confirmed that we may have arrived at a point were we shall be able to propose ways of working that can meet the true dynamics of the mind and gather unsuspected crops far beyond what was achieved more than a quarter of a century ago when the rods were used for teaching.

We do not need to spend time, as we did in the past, to make children retain arithmetical facts. Hand calculators have made all the efforts in the direction of teaching computation a waste of time. Pressing buttons and reading numerals and signs for operations is all we need as preparation to do quite complicated arithmetical computations. They take no time and, if we are careful, they are even generally correct.

To be of our time requires of us, as educators, that we know how to link the potential of the mind and the potential of our electronic technologies: computers and TV color monitors. Since I am aware that our present stage of human evolution can be defined in terms of energy, the utilization of minute amounts of energy, I have called the present "the ~~era~~ of the nothings". If we consider how to use the public fad for electronic devices in conjunction with a serious subordination of these powers to achieving maximized learning by as many people as possible, we see our task defined.

Three lines of approach on this have presented themselves to me.

First, I have invested time and money to make mathematical films by computer graphics (some of them can be seen here if you wish to). More are being contemplated.

Second, I have submitted proposals to show, on an example, how computers can be used in work with beginners so that in a very short time a considerable and valuable mathematics experience can be acquired and become functional. If these proposals can provide us the means to publicize our vision we shall extend their scope to reduce the apprenticeship of various mathematics skills to its proper duration.

Third, I am engaged in writing articles for various publications (including our own Newsletter) so that this way of relating to the powers of the computers will be examined and its merits, if any acknowledged.

Working on computers and on TV makes me more aware every day that we must all become experts (among other things) in working on "nothings". Our collective future depends on our capacity to produce a generation of minds who can synthesize what is relevant today and to offer our children effortless learnings in all fields.

Nowadays a generation gap can happen every few days, not only between people of different ages but between adults, and an accumulation of lack of understanding is not in the interest of the inhabitants of the earth. When I concentrate on the future and on the era of the "nothings", I am working at being truly of my time.

The View from Below

DAVID HAWKINS

There is a story about the late E.T. Bell, true or not I know not. His son asked, "Daddy, why do they put that plus sign on the top of churches?" Bell also remarked, in one of his popular books, that medieval theologians could be regarded as frustrated mathematicians in an age when that discipline was not fashionable. The joke is convertible. Perhaps mathematicians also are frustrated theologians.

I start this discussion with such jokes because I believe that the sources of mathematical knowledge and invention are in fact rather mysterious. I have some ideas about how such mysteries can be resolved, but I don't think the task is easy.

Let me start with a paradigm case. The Euclidean style of geometry is restricted to compass and straightedge. Working once in a fifth grade class we had introduced these ancient implements, with an initial encouragement to produce any kinds of designs. One of the things which happens, under these constraints, is a multiplicity of circles of the same radius, sometimes secondary circles centered on the circumference of the first one drawn. Two girls had in fact walked the compass around its circle and come back after six steps, almost exactly to their starting point. I happened to be near and caught the question that hovered between them: "How come?" My intervention was honest and it worked. We walked another circle with great care, and the point of the compass landed, by luck, in the hole it had started from. We ended with the hexagon dissected into equilateral triangles, and at one point Suzie said, "This (a radial line segment) is the same as that (a hexagonal edge), so it *has* to be just six."

I regard this as a paradigm case because of the implied transition from the empirical "is" to the emphatic "has to be." I assumed at the time — and still do — that Suzie had shown some flash of understanding of a kind which marks the transition from fact as empirical to fact as mathematical. I have chosen it as a paradigm case because Suzie, like Plato's slave boy,[*] could be presumed to lack any knowledge of formal geometry. Suzie did give some reason for her apparent insight, she had noticed that since the pencil compass had not been readjusted, it had produced what we would call a cluster of equilateral triangles. At the time I did no lightning-quick analysis of her apparent insight; unlike Socrates I did not proceed to elicit from her the complete steps of a formal proof. I got her to show me that she was looking at half the circle, three triangles. I looked at them later myself, quite hard. I didn't yet wish to lead too strongly, and we went on instead to further partitions of the hexagon. When now I try to imagine what I might have elicited, I am fascinated. The first obvious proof I see depends on the formal Euclidean proposition that the sum of the interior angles of a triangle is a straight angle, or that for any polygon the sum of the supplementary exterior angles, triangle *or* hexagon, is one full rotation. Is there some similar formal proof which would plausibly interpret and support Suzie's apparent mathematical insight, without imputing to her any acquaintance with high-school geometry? Since symmetry is so powerful an idea, yet accessible to visual perception, is there a direct symmetry argument here which would establish her very emphatic conclusion? I leave this, as mathematicians often say, as an "exercise for the reader," though her drawing may be suggestive. (Figure 1)

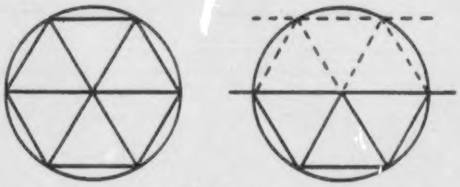

Figure 1

Maybe that particular Suzie, at that particular moment, *was* only guessing. It doesn't matter now. The story itself makes my first point.

If Suzie's insight was valid, she was already in the domain of the mathematician; in a basic sense they could meet as equals; if not, that first step was still to be taken. What then is this transition, so special to mathematics, from the merely empirical fact to the mathematical fact?

In any case Suzie had still a long way to go, even if she was momentarily in the mathematician's domain. There are myriad other facts to be recognized, of the kind I think she saw. When enough of these facts are recognized and held together they are, as a cluster, the substance of classical geometry.

For they are not independent, isolated facts, each true or false independently of the others. They are empirical facts but not what the early Wittgenstein called "atomic" facts, they are internally related to each other. One by itself, or two or three together, will lead to and perhaps require still another. If some can be noticed first as isolated empirical facts, they can still come together by some magnetic affinity, providing guides for further investigation; and the power of the process is multiplicative rather than additive in its potential rate of growth. As this clustering develops, some facts stand out as central, they lead to the recognition of many others.

This organization is what gets formalized as a deductive system. I see that as a later and quite distinctive development. It involves an abstracting, elucidation and definition of what we call idea or concepts. They are not facts, but universal terms, the elements, properties and relations involved in the perception and statement of many facts, all that potential infinity of which can be stated in this common

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[*] Plato, *The Meno*, cf. Wheeler, David, "Teaching for Discovery." *Outlook* 14 (Winter 1974). pp. 38-42.

For the Learning of Mathematics 1, 2 (November 1980)
FLM Publishing Association, Montreal, Quebec, Canada

language. Suzie's construction can be described formally in this language by a few specialized terms: points, lines, congruences, rotations, etc. These same ideas will be implicit also in the perception of many other geometrical facts and will be expressed, at first spontaneously, in stating them. The new interest in analyzing and defining such ideas leads to another sort of clustering and ordering. Some ideas emerge as primitive, central, while others can be defined in terms of them. This development puts the affinities among geometrical facts in a new light. Though endlessly diverse as facts they share a common domain and some can be transformed into others. Taking a few of them as primitives, the rest can all be demonstrated. The perception of mathematics as a deductive *system*, first clearly exemplified in Euclid's *Elements*, has been a paradigm and challenge for all subsequent mathematics, science, and philosophy. Can all knowledge be so organized? Can reason become a substitute for experience? A stubborn empiricist might insist on counting edges, faces and vertices of as many kinds of convex polyhedra as he could lay hands on, and so far notice that the number of faces always happens to be two greater than the difference between the number of edges plus the number of vertices. Euler's theorem says more, it implies that nature is not free to make exceptions, the difference *must* be just two. To the empiricist this is a kind of indignity.

At this point I have outlined two developments. The first, and most primitive, is that some kinds of empirical facts get recognized somehow as facts which must be so; like Suzie we shift them from the state of "is" to "has to be". The second stage is that clusters of such facts, facts which in one way or another seem to require each other, lead to the explication and ordering of a conceptual domain to which all these facts belong. When they are stated uniformly in the terms of this domain it becomes apparent that they are not independent of each other but are linked by bonds of implication in some orderly system. Facts now become theorems. Some theorems are chosen — as primitives sufficient to generate all the others — to be axioms.

If we wish to speak more formally about the relation between ideas and theorems we can describe these as a linkage between two domains. Ideas are related in their own domain by relations of meaning and definition. Some of these can be taken as primitive, and others defined in terms of them. Theorems are related in their own domain by relations of entailment, in which again some are primitive and others derived. But the two domains are also essentially cross-linked in one-many relations. A given idea is involved in stating several theorems, and the statement of a given theorem involves several ideas. Each domain has its own internal connective tissue, of definition in the one case and implication in the other. But the connective tissue in each domain is enriched and elucidated by cross-reference to the other.

Suzie's apparent insight started from a particular drawing which could be described as an affair of circles, lines and points. This drawing was not in her mind an *example* of anything geometrical. It was not a consequence but a starting point. But after we have developed some theorems the drawings *become* examples, in retrospect. Philosophical and pedagogical accounts of mathematics often treat examples as inessential. They are mere starting points, aids to the imaginations, etc.; but they have nothing to do with the essence of mathematics, which is entirely an affair of abstractions detached from their humbler origins, like the modern trigonometry text which has no pictures of right triangles in the unit circle. Lewis Carroll complained about this sort of thing in *Alice in Wonderland*, and he was quite right. In the View from Above examples may seem to have no essential place. In the View from Below, on the other hand, they are of the essence. They are not only vital sources of knowledge but they have a continuing and quite indispensable place all along the way.

Corresponding to any mathematical system of ideas and theorems there is therefore always also a third domain, one-many-related to the other two. It may be called the domain of examples. In the View from Below it were better called the domain of proto-examples, of originals. Any concrete example of a theorem is very likely to turn out to be an example of several more theorems. These may be closely connected in the domain of theorems, but also they may not have been seen to be, and their coincidence may suggest new theorems, or new connections between old theorems. I don't know who first looked at the drawing I reproduced here, and noticed in it the possibility of a new and immediate proof of the old pons asinorum, the Pythagorean theorem. (Figure 2)

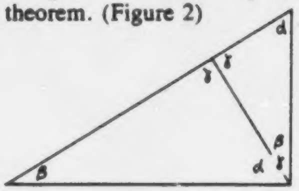

Figure 2

Instead of constructing *squares* on the sides of the right triangle it makes use of the three similar *triangles*, two of which together "have to be" equal in area to the third. The famous theorem is then seen as a direct consequence of the more general fact that the areas of arbitrary similar figures are proportional to the squares of any corresponding linear dimensions — of the flatness of space.

In their domain examples or originals are directly related to each other by similarities and differences, or by part-whole relations, etc. They come also to be indirectly related to each other through the domains of ideas or of theorems. Thus the cube and octahedron are indirectly related by the fact that the description of the one is transformed into the description of the other by simply interchanging the words "faces" and "vertices" and this duality immediately suggests a way of constructing the one from the other; or vice versa.

I should like at this point to acknowledge a substantial debt to the work of Edwina Michener,* from which I have taken the above three-fold partitioning of elements in the structure of mathematics. Her clear definition and discussion of these three domains provides I think a most useful frame for our discussions. She demonstrates this usefulness in several ways, illustrated from several fields of mathematics, elementary and more advanced. A key aspect of her work is to elucidate the nature of mathematical *understanding*. Rather however than trying to summarize her discussion I shall continue on my own track, but with that same concern for the meaning and importance of the active verb *to understand*.

I grew up with a somewhat rebellious acceptance of the notion that mathematics was by its nature a highly sequential affair. I gained this impression first from the format of courses and textbooks, and later from some study of formal logic and the foundations. The pecking order of pure and applied math was then the order of the day, and the dominant positivist trend in philosophy, following some traditions of research in mathematics itself, was to declare that what was real was empirical and what was rational was — however elegantly — empty. This emptiness was to be demonstrated by the exhibition of completely formalized mathematical systems bled of all conceptual or factual content, yet with no loss of formal coherence. In such a demonstration the theorem domain could be disconnected from those of ideas and examples, by depriving its formulae of any taint of vulgar meaning.

This movement can be considered from two points of view. As a special movement *within* logic and mathematics it has been part of a whole new investigation of a meta-formal character concerning the logical nature of mathematical system, dealing with problems of consistency, completeness, the relation of axiom systems to models, etc. As a *special* mathematical discipline it clearly fits the general framework of Michener's analysis; it has its own genres of theorems, concepts, its originals being the mathematical systems. But the general philosophy I have alluded to, which for its own ends identified the theorem-structure of formulae with the whole of mathematics and left other aspects uncultivated, no longer seems very fresh or intriguing. Yet its legacy is still with us, the image of an essential part of mathematics mistaken for the whole.

Having been strongly influenced by that philosophical movement, though never embracing it, I tend to be inhibited in any outright opposition to it. Such opposition seems to embrace some allegedly outmoded philosophy of Platonic, Aristotelian or Spinozistic rationalism. Yet the desire to oppose has grown steadily. Apart from my own roots in those older philosophical traditions, I have been strengthened in this desire by endless professional curiosity about the thought habits and commitments of physicists, biologists, economists, children, and mathematicians. Having been a student of the last two I have found I frequently came back to mathematics from a new point of view, usually mis-named *applied*. Whenever this happens I find that though the formal academic backtround has helped, I frequently arrive, after some struggle, to a sense that I have not merely *applied* some mathematics, but in the process, and with help, invented or reinvented or extended it along shortened pathways quite different from those I have been taught. Indeed I have often reinvented the wheel, but have come in the process to realize that in such matters — the wheel is a good example — no one has yet quite said the last word on the subject or its implications. These pathways typically start from "applications" but also typically bring close together ideas and theorems which have been treated in quite different parts of the book or in different books. I have begun to get a clear sense that the mathematical territory was not so much like the directed graphs or trees of the theorem-space, but much more like a network in which one could go from A to B by a variety of routes and back again along still others.

Tree structures there are, but they are so interestingly cross-indexed by analogy and reference that no simple metric of distance or closeness, such as that represented by the numbering of theorems in a book, or even the partial ordering of a theorem tree, seems appropriate. One is involved in a network somewhat like a map of airline routes, and like the airlines it brings many things close together. The map is, of course, a high-order abstract. To know the landscape and the culture one must in person get there. I have also realized, however, that the image of the net needs major constraints. If every node is connected directly to every other node one is informationally swamped in finding the relevant connections. Nodes must be graded by importance, by generality of relevance. Nodes are not all of the same kind, moreover, nor are their connections. What is first seen as a connection, for example the idea of a shared property relating different examples, can thereby itself become a node connecting to other ideas, sometimes by shared examples. This sort of duality transformation seemed to blur the image of the net. Michener's representation helps; it allows three planes or spaces in each of which the items and relations are of the same kind, but with cross-referencing by specific dual relations to items in the other two. I think then the image becomes sharp again.

Michener is concerned primarily to use the net as a framework for improving our understanding of mathematical *understanding*, and thus for a description of the personal qualities which we call fluency, resourcefulness, competence. Understanding of course implies knowledge, and knowledge implies subject matter, some independent domain of fact which is just there, incompletely known. The sum $\zeta(2) = \sum_{n=1}^{\infty} {}^1/_{n^2}$ is ${}^{\pi^2}/_6$. I know it, I have even followed and agreed to a proof, but I emphatically do not understand the result; others I hope have a better understanding. Nobody knows what to say about $\zeta(3)$, except that as has recently been shown, it is irrational. I think I understand something about such sums, can transform them, can disentangle connections among them, but the above trigonometric link eludes me. Understanding implies knowledge, though I think it need not imply formal proof. Indeed understanding is often the guide to the invention of a proof, or a better one. I think most of us know enough about the counting numbers to see why the unique factorization theorem is true, and perhaps why it won't be true in some otherwise analogous number fields. Maybe if I were hard-pressed I could invent a proof.

If the goal of mathematics education were really to work for active understanding on the part of students, what would the consequences be? Michener has experimented success-

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*Michener, Edwina Rissland, *The Structure of Mathematics*, M.I.T. Artificial Intelligence Laboratory, A.I. Technical Report No. 472, August, 1978.

fully with the deliberate and conscious use of her scheme among university math students. No doubt many good teachers do something like that implicitly.

With the emphasis on understanding, on fluency, what does the picture look like for the early years? I cannot give a research report, but only suggest the outcome of a good many years of elementary school work, some with children, some with their teachers. One results is to see a need to modify or amplify the foregoing account. It is one thing to try to extend mathematical understanding among those who already have some formal education, but perhaps quite another to give access to it in the first place? I believe there are two complementary answers. One is from Plato's myth of reminiscence, that children *already* have some mathematical knowledge and understanding, and that we can learn to recognize this and as teachers, resonate with it. The other is that this understanding is typically implicit, context-dependent, and often hardly available for expression in the verbal or notational mode. Henry James says it with characteristic rigor in the preface to *What Maisie Knew*: "Small children have many more perception than they have terms to translate them; their vision is at any moment much richer, their apprehension even constantly stronger, than their prompt, their at all producible, vocabulary." In building bridges for communicating with children one needs therefore to learn (in part by reminiscence?) their characteristic ways of thinking, and this I claim involves a genuine extension of our own mathematical thinking.

This bridge-building also requires, as a result, some qualifications of Michener's scheme.* For children, theorems are not yet theorems, concepts are not yet concepts, examples, above all, are not yet examples. To say this is a less Zenish language, they can and will often employ patterns of recognition and thought, yet be unable to scrutinize them, or communicate about them in anything resembling adult language. What one discovers, through trial and error in teaching, is that their powers of communication are immensely greater when they and the teacher are in the immediate company of the concrete situations, the originals, out of which their understanding has manifested itself. Piaget talks about this phenomena as reflecting a concrete operational stage of thought, which in one sense it does. But this is often taken to mean that children are incapable of having and using high-order abstraction. This is another claim altogether, and I think a false one. Suzie's symmetry argument, as I call it, is rather deep, though she could not at all spell it out. She can only communicate it to me pointing to elements of the diagram we have produced, using gestures and demonstrative pronouns, appealing to *my* visual perceptions. I can't quite partition her insight into theorem, concept, and example, though I can — after some effort — validate it that way.

So for present purposes, as I said earlier, the term *example* is ill-chosen. An example becomes an example only after it is an example of something *previously* named and recognized in the domains of theorem and concept. What is seen is a concrete particular seen *as* something understandable. *That* something is still implicit. For me it is an example, for her it is still a concrete — though somehow pleasing — particular, an intriguing fact. I led her along the pathway of my kind of analysis as far as I dared, hoping to help her build bridges into an adult world.

A mathematician only extends his own understanding by active search, by being personally in charge. He can accept help from talk and print and can follow an argument if he has already shared some of its turf. But he alone develops his understanding. How does this translate into the childhood context? The major change, I believe, is that we must learn to share the childhood turf. A part of this can be at times almost adult a thing of paper, pencil, of books, even at times I suppose of workbooks. For most children, most of the time, the turf is different. It is the world of concrete experience, presentational rather than linguistically representational. In this world the activity which leads to understanding is not yet separated from overt activity, it is directly perceptual and, a term of Jerome Bruner's, enactive.

In our own work, for such reasons, we have made ample use of the now-commercial concrete math materials, and added others as we were bright enough to think of them; pegboard and golf tees for lattices and graphs, for Mary Boole's curve stitchery; many of several shapes of geometrical tiles for tessellations and growth patterns, various looms for weaving (lit., com-plications), marbles for 3-D patterns, card for making polyhedra, poker chips for graphs and patterns, etc.

An interesting fact about the commercial materials is that often their most appealing uses are those not intended by the designers. Thus the so-called Cuisenaire rods were intended primarily for arithmetic, but I have never seen children first use them for that purpose. The intended use is representational. They were conceived I think as examples of the little number facts of early arithmetic, and their 2-D and 3-D extensions. They were conceived, in short, as new tools for the didactic teaching of arithmetic. What they get spontaneously used for however, is presentational rather than representational; they are fine for building "complicated" and elegant patterns, some of which may happen along the way to raise some very nice questions of arithmetic or geometry. We assign a sort of figure of merit to these commercial materials, the ratio of their unintended usefulness to that intended. In these unintended uses children show you some of the rich turf you have to learn. Natural materials — such as mud-cracks and growth patterns — are often better.

It has been a great help and moral support to me in this work to realize that much of early Greek geometry and arithmetic was developed by the use of Cuisenaire rods. John Trivett and his students found they could represent the sum of successive squares by a rectangle built of rods, but only if they combined three such sequences together. He told me that he then really *understood* why that algebraic formula had a six in the denominator! They had in fact rediscovered a theorem of the Pythagoreans, who also extended the method to find the sum of cubes. I gave these

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*to which I hope she would agree, cf. Michener, E., "Understanding Understanding Mathematics," *Cognitive Science* 2, 361-383 (1978), and for comparison: Hawkins, D. "Understanding the Understanding of Children," *American Journal of Diseases of Children*, vol 119 (Nov. 1967), also in *The Informed Vision*, New York: Agathon Press, 1974.

problems once to a class of students in an Analysis course, and only four solved them all, using analytical methods. I regard such students as under-privileged.

When you are first inventing geometry (or is this number theory?) you don't use standard methods. You have to develop your understanding.

One of the results of working in this style is that you get into mathematics, not just computational routine. The computation comes along, and quite a bit of it could be called practice. In finding tetrahedral numbers, the sum of successive triangular numbers, you can do a lot of sums before you see the pattern. And when you encounter those sums again in random walk investigations, also after much numerical calculation you are on the edge of a deeper understanding. I emphasize the computational aspect because it is, among other things, of some importance; but also because if you thought someone on the school board would regard your work as time-wasting play, you could point with pride, inter alia, to the number skills.

My purpose has been to try to map a useful and plausible account of the structure of adult mathematics into the childhood milieu. If that doesn't work, something is wrong with the account itself. If it does work, it still may qualify the description of the structure of mathematics and mathematical understanding. If my interpretation of the story of Suzie — and we all know other such stories — is correct, we should look more carefully at the ways in which the domain of mathematics, which in some essential sense is discursive, symbolic, digital in its mode of expression, nevertheless is linked to that which is perceptual, presentational, implicit. Such a linkage is required, I suggest, by any account of the historical origins of mathematics *or* of its successful pedagogy. I think it is also needed in any account of later major developments within mathematics itself. If one traces the origins — the originals — of such development, one finds that they very often turn upon some fresh success in discursive explication of the perceptual and intuitive. Greek geometry surely depended on such an explication. Its axioms, once explicated, were "evident." The parallel postulate in particular is an explication of the perceptual symmetries of certain lines and angles, themselves defined by symmetries. Archimedes introduced novelty by his derivation of the hidden symmetry of the law of moments from the intuitive symmetry of the equal-arm balance, and used this as a new tool of investigation in geometry. Bernoulli appealed to the intuitive symmetries of gambling devices, and from this derived his famous theorem, probably the first major mathematical step beyond the practical lore of gamblers. Connected to the ideas of groups and invariance, the symmetry principle became the basis for whole new developments in geometry and also in theoretical physics. Still more recently it has legitimated the use of probability theory within number theory and in geometry itself.

If one takes such a series of examples of the way in which our understanding can sometimes be mathematized, one is — or can be — tempted to return to those older rationalistic philosophies which I have mentioned earlier. They reserved a place, at least, for the notion that at any given stage in its career the mind possesses some furniture which is in no obvious way simply the outcome of empirical induction, but which it can bring *to* any new experience as a means for reducing that experience to order, of reducing the apparent redundancy of experience. If one does not like the classical rationalism with their appeal to innate ideas, one can try the move initiated by Kant and treated developmentally by Piaget.

If one is temperamentally suspicious of all such grand philosophical moves, however, there is another track to try to follow, more modest and in its own way empirical. The one example I have suggested is a kind of examination, from historical and contemporary sources, of the ways in which arguments from symmetry have contributed to the mathematizing of otherwise only empirical subject matter. I have mentioned examples from Suzie (and Euclid), from Archimedes, from Bernoulli and those who followed, from the Erlangen program, and — most recent — from the way in which Buffon's ideas of geometrical probability have been taken out of the limbo of mere empiricism by the insight, first apparently voiced by Poincaré, that a mathematically adequate definition of geometrical probability emerges from the choice of that measure which is invariant up to an appropriate group of geometrical transformations. The development of this program has contributed to new extensions of geometry itself, as well as to many practical applications in stereometry, etc.

I could mention also the many fascinating examples of symmetry and invariance which have been first postulated, in an apparently high-handed a priori fashion, by theoretical physicists, and often enough (though not in every case!) empirically confirmed. Rather recently, it seems, good formal arguments have been developed which derive the classical conservation laws from the symmetries of space. I don't understand these arguments yet, but they seem at first sight to be legerdemain at variance with good old-fashioned empiricism. At any rate and in the meantime this whole history seems to suggest something important about the process of mathematizing, at least one long and tough thread of continuity between what Suzie knew and the most recent higher development of some parts of mathematics and physics. I don't know how to assess this, though it would have delighted the hearts of the old rationalists. If they are wrong we need to find some adequate account of such matters, one which among other things might be pedagogically important.

I think in fact the old rationalists were wrong, though the standard empiricism is wrong too. Even along the one line of continuity I have suggested there are incursions of novelty into the development of geometry, modification, extensions of, and attacks upon, preconception. The most cherished intuitions, once axiomatized, are open to revision. This is very far from saying they are arbitrary; the intuitive symmetries can be played with, and new pathways explored, the old symmetries subtly modified, as in non-Euclidean geometry. Some "firsts" are so important we should learn to be playful about them — but not before we understand their power.

CMESG/GCEDM
Working Group A

THE TEACHING OF CALCULUS AND ANALYSIS

Leader: Peter Taylor

Calculus text books tend to be all much the same. They are technique - and type-problem - oriented, and aim to give the student competence with a large well-defined body of results. They present these results in a highly systematic manner which tends to regiment the student.

The original purpose behind this working group was to design written materials for the teaching of analysis which would be much more flexible than the standard texts and encourage the student to confront and explore some of the mathematical ideas and get involved himself in the formulation of problems and results. We use the word "analysis" rather than "calculus" because the goal of the first year course should be to endow the student with a variety of weapons for the analysis of functions, of which the standard tools of the differential and integral calculus are important examples, but not always the most appropriate. Thus, often, a good question to ask is, let's see how much we can say without using calculus. With this approach one becomes more sensitive to the power of calculus.

The group was large and diverse. This created some interesting difficulties, gave us some new insights and certainly retarded our progress towards the above lofty goal. Right away it became clear that there were widely different opinions concerning the way in which our group should spend its time. At one pole there was expressed the importance of bearing down on some important technique or idea in calculus (eg the notion of limit) and discussing how one should present it (should one use ϵ, δ, if so how?). At the other pole was the desire to deal with the general question of how to get the student to explore the ideas himself and to find problems which would encourage him to do so with the ideas of analysis. These points of view may not be as far apart as they at first appear, but a lot of time was (necessarily, I think) spent trying to find the right balance. It is an old problem and closely related to the question of what teacher training should consist of.

In the end we took a sequence of topics in the curriculum and bore down on them in an attempt to find problems which would get the central idea across, encourage the students to explore, and promote some facility with the techniques which have arisen from the idea. In the appendices we present some of the problems which were discussed.

We discussed the need for periodic communication between different mathematics departments in Canada on the teaching of analysis, including regular exchanges of problems and exams. More demographic information should be available: what sort of students are taking these courses now, what are their needs, how does the course as it is now presented interact with their future studies and work? We suspect that far too little attention is paid to these questions, and we have arranged for the collection of some materials between now and our next meeting which may provide some answers.

Report on Working Group A

APPENDICES

A	Preliminary discussion papers	Peter Taylor
B	Does calculus have a gist?	Ralph Staal
C	Applications of mathematics	Ralph Staal
D	Reflexion sur l'enseignement du calcul differentiel et integrel au niveau collégial, à partir d'applications	
E	List of participants	

APPENDIX A

GEOMETRY AND ALGEBRA

Geometry is the world of pictures. It casts a light which illuminates our work and gives it warmth and beauty. It guides us in our search for what is true and suggests new directions which might be fruitful and new problems which might be interesting. It provides a framework in which we can organize our knowledge so that we can recall it when needed and apply it to our purposes. Its power is the power of imagination. But like all lively creatures, it can deceive.

Algebra is the world of symbols. The light it casts is hard and sure. It allows us to pin our ideas down and make our truths precise. Like geometry it has its patterns and intuitions, but they are more difficult to perceive and work with. Its manipulations demand experience and care; its calculations can be brutal. Its power is the power of precision.

Each world has its own language, and our main ideas will be formulated and explored in each of the two languages. It is most important that we learn how to translate from one language to the other. Only then can we pass easily from one world to the other and use the grammer of one to gain insights into the structure of the other.

3. AN ESTIMATION PROBLEM

3.1 PROBLEM Once upon a time I was discussing metrication with one of my calculus students, Chris. Chris was a keen runner and was explaining to me how the records for the yard and mile distances were already out of line with the metric records.

"The major meets are almost all metric now. The metric records are already better than their yard counterparts."

"How could you see that? The metric events are of different lengths than the yard events."

"You'd have to find a way of comparing them Perhaps if you plotted the current world record time of an event against its length, you'd see the metric events falling into a certain pattern and the yard events just off the pattern."

"That sounds interesting. I'd like to see such a graph. I bet the difference in the patterns would be very slight. Perhaps hard to detect."

"You may be right. The difference wouldn't be uniform either. For example the mile, because of its long tradition, is probably just as frequently contested as its metric counterpart, the 1500, possibly even with more enthusiasm."

"Ah, so it should be right in the metric pattern or possibly even better?"

"I should think so. How would you find out? Is there a method for doing things like that?"

"I don't really know, but it sounds like an excellent problem. I would just start by looking at the data and hope some ideas come along."

"I have the metric records at home."

"Why don't you go and look at them and see what you can do. In fact, take the following problem. Look only at the metric data. On the basis of any patterns you can perceive in this data, produce upper and lower bounds for the record time for the mile, assuming that it conforms to the patterns. See how close you can make these upper and lower bounds. Then, after you've done that, we'll take the actual record time for the mile and see if it lies between these bounds."

"That sounds like fun."

"I'll expect a report tomorrow."

SOLUTION Chris appeared the following morning with tables and graphs.

"Well," he exclaimed,"I had quite a night, Wait till you see this."

METRIC WORLD RECORDS (Aug.76)

x meters	T secs	Year set
100	9.96	68
200	19.81	71
400	43.9	68
800	103.5	76
1000	133.9	74
1500	212.2	74
2000	291.4	76
3000	455.2	74
5000	793.0	72
10000	1650.8	73
20000	3444.2	76
25000	4456.8	75
30000	5490.4	70

"I'm quite excited myself", I assured him quietly.

"You were wanting upper and lower bounds based on metric distances, for the record time for the mile. Well, the obvious thing to do first is to take the 1500 meter time, calculate the average speed for that race and assume that the mile (1 mile = 1609.344 meters) could be run at that speed. The time you calculate for the mile should underestimate the actual record. On the other hand if you do the same thing for the 2000 meter time, you should overestimate the mile time."

"Good, what did you get?"

"Here is the metric data in this table taken from the current Guiness Book of Records. Using the 1500 meter pace I get a time for the mile of 227.67 secs. and the 2000 meter pace gives 234.48 secs. So if the mile time is to conform to the metric pattern, it must certainly lie between these two numbers."

"Right", I said, "a difference of about 7 seconds. Not all that good."

"I agree. But now watch this. The next thing I did was to graph the data to see whether I could see any patterns. Here's the graph I got. You can see that it's concave-up, which is just

Time for the mile assuming 1500 m pace used

$$t = \frac{1609.344}{1500} \cdot 212.2 = 227.7$$

Now assume 2000 m pace used

$$t = \frac{1609.344}{2000} \cdot 291.4 = 234.5$$

what you'd expect."

"Why?"

"Why what?"

"Why would I expect the graph to be concave-up?"

"Uh, because longer races are run at lower speeds."

"That is the same as concave-up for this graph?"

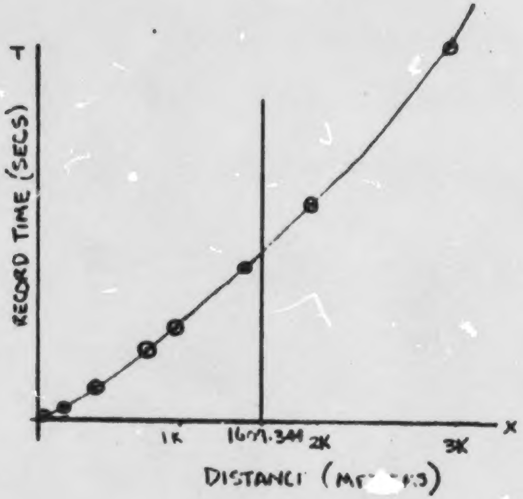
"I think so."

Well, I'll leave you to check that out. Anyway I can certainly see that the data points are concave-up."

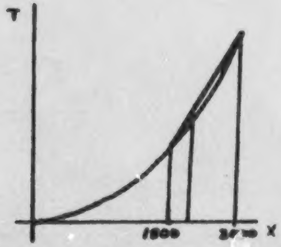
"Alright. Now suppose we let T(x) be the hypothetical record time for the x meter race, assuming that it was a regular event at all metric track meets. Then the graph of T(x) is a smooth curve which passes through these data points."

"And our job is to estimate T(1609.344)."

"Right. Now we have just noticed that, according to the pattern perceived in the data points, the graph of T should be concave-up. This means that any secant drawn between two points on the graph, will lie above, the graph at intermediate points. In particular the secant between 1500 and 2000 will lie above the graph at 1609.344. Now we can use similar triangles to



a plot of the data points with the graph of the hypothetical function T(x) passing through them. Our job is to estimate the height at which the curve cuts the vertical line x = 1609.344



Since the graph is concave-up the secant on [1500, 2000] will lie above the graph at intermediate points. The concavity is exaggerated in this and subsequent pictures.

calculate the height of this secant at x = 1609.344. We get a height of 229.52 seconds. This must be greater than T(1609.344)."

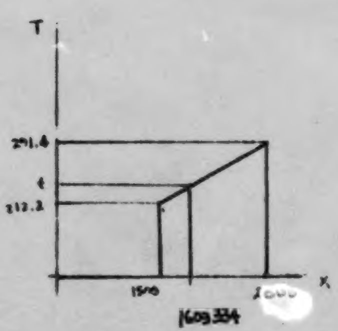
"Excellent. A considerable improvement on the previous upper bound of 234.48. You've got the mile time pinned down to within 2 seconds. Can you see anything else to do?"

"Well, yes I can, although it took me a long time to see this. I got inspired around midnight. You can actually use the concavity of the graph again to improve the lower bound. The idea is this. Take the secant between 1000 and 1500. Since the graph is concave-up, it will lie above the graph at intermediate points, but below the graph if I extend it in either direction. In particular it will intersect the vertical line x = 1609.344 below the graph."

"Ah, very good."

Chris was pleased. "So," he continued with a flourish, "Using similar triangles again, we calculate the intersection height to be t = 229.32, a new and formidable lower bound."

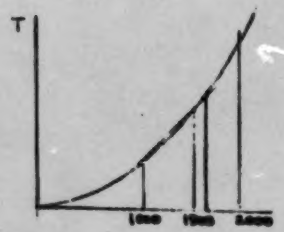
"My goodness," I said, "What was the new upper bound? 229.52 ? That is a gap of 2/10 of a second."



Using similar triangles to calculate the height of the secant at x = 1609.344

$$\frac{t-212.2}{291.4-212.2} = \frac{1609.344-1500}{2000-1500}$$

Gives t = 229.52



Since the graph is concave up, the secant on [1000, 1500] will, when extended lie below the graph

"Pretty good eh?"

"So in order for the mile time to conform to this simple concave-up pattern of the metric data, it has to lie inside an interval of length 2/10 sec. I'm amazed. I didn't think the requirements would turn out to be so stiff. What is the record time for the mile anyway?"

"I have it right here. Are you ready?"

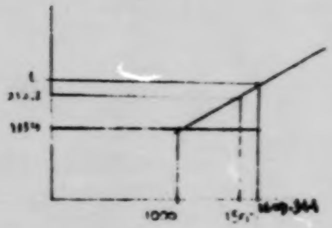
"Ready."

"As of August 1976, set by John Walker of New Zealand at Gothenburg, Sweden on Aug.12, 1975: 3 minutes, 49.4 seconds."

"Which is ...."

"229.4 seconds."

"Wow."



$$\frac{t - 133.9}{212.2 - 133.9} = \frac{1609.344 - 1000}{1500 - 1000}$$

$$t = 229.32$$

3.2 PROBLEM Chris was very proud of himself. "Do I get 10/10?" he said.

"I guess so," I said, "Well, actually not quite yet. You have to finish the job. We have a couple of ends to tie up. You recall the first set of bounds which you obtained with an average speed argument? I want to see those estimates on your graph.

I want a geometric interpretation of your argument. I want to know precisely what property of the graph is used to make the argument. I don't happen to think its quite the concave-up property, because that is what you in fact used to get the second set of bounds. Secondly, going the other way, I want a physical interpretation of the concave-up property of the graph. If I hadn't seen the points plotted, is there anything about their physical interpretation which would allow me to predict that they would follow a concave-up pattern? So, in summary, I want the physical/ geometric analogy completely pinned down."

He nodded ever so slightly and left my office.

SOLUTION "How did you make out?" I said when Chris appeared the next morning.

"Not too badly. What I've done, I think I've done properly. But there's one thing I just can't seem to get."

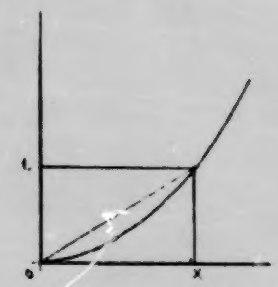
"Well, let's see what you've done."

"The first thing is to find the geometric interpretation of the fact that the average velocity for the race of x meters should decrease as x increases. Now if we draw the secant from the origin to the point on the T-graph above x , we get a line whose slope is elapsed time over distance run. Thus this slope is 1/v(x) where v(x) is the average velocity for race x . If v(x) is to decrease as x increases, 1/v(x) must increase. That is the slope of this secant increases with x .

"Indeed," he went on, "to put the result neatly let us make a definition. Suppose f is a function defined on some interval [a,b]. We will say f has the increasing secant property from a if the secant to the graph of f on [a,x] has slope which increases with x for x > a . Then to say that the longer the race the lower the average velocity is exactly to say that the function T has the increasing secant property from 0."

"Very good," I said. "So geometrically, it is what you call the increasing secant property which allows you to obtain the first set of bounds."

"Right. I've drawn a picture with all our bounds displayed so we can see how they are determined. The second and final pair bounds is obtained from the secants on [1500,2000] and

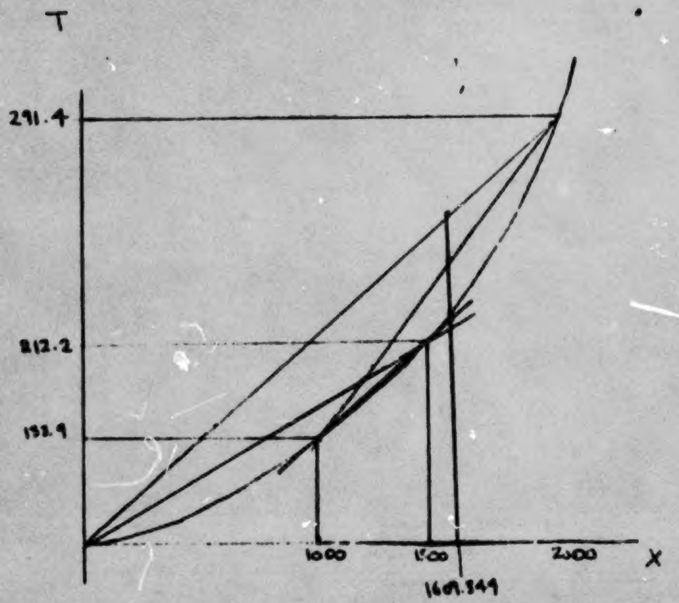


Graph of the function T

If the race of length x is run in time t, then the slope of the above secant is $t/x = 1/V(x)$ where $V(x)$ is the average velocity for this race.

As x increases we expect on physical grounds, that $V(x)$ should decrease. Thus the slope of this secant will increase.

Thus T has the increasing secant property from 0.



Graphical interpretation of the estimates. They are, from the bottom up.

1) 227.7 assumes the 1500 m. pace is used—an underestimate.
2) 229.32 linear extrapolation from [1000,1500]. An underestimate since the curve is concave-up.
3) the "correct" value of T.
4) 229.52 linear interpolation value. An overestimate since the curve is concave-up.
5) 234.5 assumes the 2000 m. pace is used—an overestimate.

[1000,1500]. The argument for them requires the graph to be concave-up."

"Ah, ha. And that is not the same as the increasing secant property?"

"No it's not. Concave-up is stronger than increasing secant property. That is the result I'm pleased about. I can show that any function f which is concave-up on an interval [a,b] has the increasing secant property from a . But the converse is false."

"Ah, that is very interesting. You have an example of a function f on some interval [a,b] which has the increasing secant property from a , but is not concave-up?"

"That is right."

"Well you hang onto it. It sounds like an excellent problem. I'll assign it to the class on Friday. You'll be a step ahead of the others."

"Now there's still one more thing. You wanted a physical interpretation of the concave-up property of T , and from that, hopefully, a reason why we might have expected the graph to be concave-up before we saw it."

"Right. Any ideas?"

"Nope. I'm sure it must be easy. But I just can't see it."

"Well, if it's any comfort to you, I can't see it either."

"Really?"

"Really. Perhaps it's not so easy after all."

"Well, I'll be dammed."

"It is a curious problem. It really should be easy. Perhaps I'll assign it to the class as well. See what happens."

3.3 Women have a 60 meter track event. The world record for this race is 7.2 seconds set and reset by many different women between 1960 and 1975. Now the 100 meter women's record is 11.01 secs set in 1976. Observe that the 100 is run at a *greater* average speed than the 60, differing in this regard from the pattern observed in the longer races. The reason of course is that the stationary start affects the shorter race proportionately more than the longer race, and no appreciable factor of tiring operates between the 60 and the 100 meter races.

(a) Here's your problem.

Assuming that a woman in good condition can run 200 meters at top speed, use the 60 and 100 meter data to calculate a record time for the women's 200 meter race on a straight track.

[Note: the women's 200 meter is usually run with a turn. The record time for this event is 22.21 (1974) which is more than twice the 100 time. The effect of the turn is greater than the effect of the start. One can see the same situation in the men's events. The 200 meter record I have given you is the event with a turn. The 200 meter race on a straight track is also run, although infrequently. The world record is 19.5 (1966) which is, as expected, less than twice the 100 meter record of 9.96. Another problem than, similar to the one I've just given you, is to calculate a record time for a men's 60 meter event, from the data for the 100 meter and the 200 meter (straight).]

[Hint. The way to set the problem up algebraically is to suppose that all these short races (called "sprints") are run according to the following pattern: a certain distance x_0 is used to attain maximum speed. It takes a certain time t_0 to cover this. The remaining distance is run at top speed v_0 . These three "unknowns" are the same for all sprints. Now the data that you have been given allows you to calculate v_0 and with this you can solve the problem.]

(b) Having solved (a) you will have found the top speed v_0 for a woman. Use the men's data contained in the above note to calculate a top speed for a man.

The answers for top speeds are as follows:

Women 10.50 m/sec

Men 10.48 m/sec

This result is anomalous as we expect the male top speed to be the greater. (?) The main reason this happens is, I think, the fact that we used the 100 m. and the 60 m. races to do the female calculation and the 200 m. and the 100 m. for the male. Perhaps our assumption that a male can run at top speed for 200 m. is unreasonable.

(c) This problem is concerned with the type of information that would be needed in order to deduce values for x_0 and t_0 (which you cannot do with the information you have so far). You may know already from calculus or physics that a person accelerating from rest at constant rate a , travels a distance at^2 in time t . Assuming that at the start of a sprint a woman accelerates at a constant rate until she hits top speed, calculate x_0 and t_0 from the data given. [This assumption is physically highly implausible, but it makes for a good mathematical problem, and it indicates the type of information that is required to find these numbers.]

3.4 According to the Guiness Book of Records (1977) the World record for 10 miles is 2757.2 secs. (1975). Show that this time is much slower than our metric data would predict. [1 mile = 1.609344 km. With the emphasis now almost entirely on metric distances, it is unlikely that this event will ever "catch up" to the metric pattern.]

3.5 What calculations would you make to show algebraically that the given data points for the function of 3.1 are indeed concave-up? Use a picture to illustrate your answer.

3.6 a) Suppose f is a function defined on [a,b]. Show that if the graph of f is concave-up, then it has the increasing secant property from a .

b) Convince me, by producing an example, that the converse is false.

*c) Can you think of a physical reason why the graph of T(x) of Problems 3.1 and 3.2 should be concave-up?

Approximation

The fundamental idea of functional (local) analysis is that of approximation. Frequently, calculations are too difficult or impossible to make exactly and must be made approximately. How can we obtain good approximations, and how can we obtain good bounds on our errors?

1. The fundamental idea of differential calculus is that smooth functions behave locally like straight-line functions. This idea provides an obvious, and geometrically compelling, way to obtain local approximations of functions. The standard text-book treatment of this technique exemplifies everything that is bad about calculus texts. A typical exercise requires the student to calculate a first-order approximation of $\sqrt{9.02}$. He looks the formula up in the example which preceded the exercise, and with a little luck (the example unfortunately calculated $\sqrt[3]{8.03}$) he gets the answer. The total ineffectiveness of this approach was driven home to me when I discovered that students who could solve the above exercise reliably, could not in fact handle problems of the following type.

Problem. I am interested in the behaviour of the square-root function $f(x) = \sqrt{x}$ in a neighbourhood of $x = 9$. In particular I want an approximation to $\sqrt{9.02}$. Here's an important idea. Straight line functions are very easy to calculate with. So replace $f(x)$ with the best straight line function $g(x)$ in a neighbourhood of $x = 3$. (Choose graph (g) to be tangent to graph (f) at $x = 3$.) Calculate $g(9.02)$ and use this an approximation of $f(9.02)$. Use a simple property of graph (f) to decide whether your approximation is high or low. Use some simple geometry to get a bound on your error.

A student who has mastered (or is mastering) the basic geometric idea of derivative (slope) should be able to (and required to) handle the above problem. Once the idea is mastered everything else follows, for example, the chain rule.

2. Recursive equations $x_n = F(x_{n-1})$ often arise in practice. Size at time n is determined by size at time n-1 . What happens over time if we start at some value x_0 ? The student should be asked to see what happens geometrically. An idea of great practical importance here is that of stability. Suppose x^* is an equilibrium point $x^* = F(x^*)$. What happens if we start at $x^*+\varepsilon$ for small ε ? The student should explore this geometrically and with a calculator for particular examples. He should perceive the almost linear convergence with the calculator. He should be led, with his straight-line approximation idea, to the standard theorem. (Try $F(x)=(x^2+3)/4$, $F(x) = x^2$, $F(x) = \cos x$ with various starting points.)

3. Here is an excellent problem using the same ideas. My friend has the following method of finding a fixed point of a function F(x) . He programs his calculator to compute values of F(x) and then, starting near the fixed point he is looking for, he runs his program again and again (calculating F(x), F(F(x)), F(F(F(x))) etc.) until the display stops changing. He then has (an approximation to) his fixed point. Will his method always work? What's happening geometrically? Suppose you can calculate values of F'(x) as well as F(x) . Can you think of a method of getting faster convergence?

4. How does the largest (real) root of the polynomial $x^3-x^2+\alpha x-2$ depend on the parameter α in a neighbourhood of $\alpha = 2$? [The student should do explorations with a calculator and then come up with a theorem.]

CHAPTER 3

Qualitative Behaviour (Global analysis)

Often it is important to obtain a general idea of the behaviour of a function over a sizeable interval. Some of the tools that are useful here are critical point analysis, concavity, and asymptotic behaviour. It is important for the student to have a good stock of standard examples whose qualitative behaviour he understands [polynomials, rational functions, trig functions, exp and log, square root, etc.]

There is far too little attention paid to results of this nature in standard calculus texts. Often there are little more than a few graphing problems. In the following I present a few reasonable problems. I should like many more.

11.2 CLASS PROBLEM Here are some examples of qualitative results. Decide, on the basis of your geometric experience whether or not they are true. If they are false, draw the graph of a counter-example. If they are true, ask whether or not you can find a proof. By the end of this book you will have enough machinery to prove them all (those that are true!), so come back to this problem from time to time. Assume all functions are continuous and differentiable.

1. A function defined on all of R which is concave-down and not constant must assume negative values.
2. If f is concave-up on $[0,\infty)$ and not constant, then it is unbounded.
3. If f is concave-down and bounded on $[0,\infty)$ then it must be asymptotic to a horizontal straight line.

4. Suppose $f(0)=g(0)=0$ and $f'(0)>g'(0)\geq 0$. If f is strictly concave-down and g is strictly concave-up then the equation $f(x)=g(x)$ must have a non-zero root.

5. If f and f'' always have opposite signs (i.e. $f''/f<0$ if $f\neq 0$) on $[0,\infty)$ then f has infinitely many zeros.

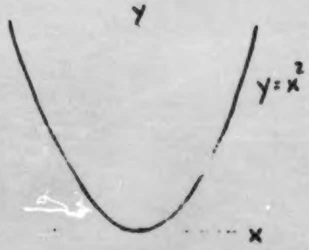
6. Same as 5 with $[0,\infty)$ replaced by R .

7. Same as 5 with $f''/f<0$ replaced by $f''/f\leq -1$. [Can you think of a function with f''/f always equal to -1?]

*8. If $f\neq 0$ and $f'(x)/f(x)\leq -1$ whenever $f(x)\neq 0$, then f has no zeros.

11.3 CLASS PROBLEM Qualitative classification of polynomials. Let us suppose we are interested in classifying the graphs of polynomials of a certain degree according to shape. So we are not interested in where the graph is located with respect to the axis but only in its shape. Thus we are not interested in the sign of f, or its roots, but in its slope and concavity: the sign and roots of f' and f'' . Our objective is classify polynomials by these characteristics: how many different types are there and how do you tell to which type a given polynomial belongs.

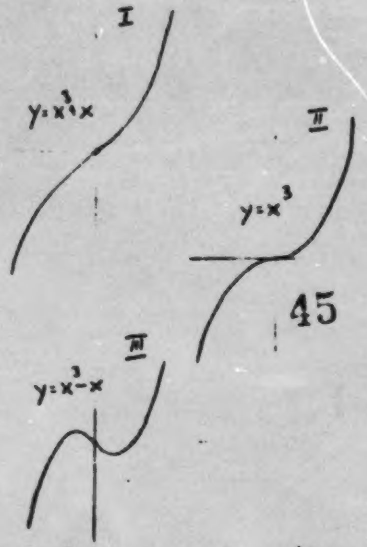
A polynomial is monic if the coefficient of the leading (i.e. highest order) term is one. We will restrict our considerations to monic polynomials. All other polynomials are simply multiples (positive or negative) of monic ones.



All monic quadratic polynomials have graphs which are qualitatively like this one.

Let us first look at degree 2: give a qualitative classification of monic quadratic polynomials. That is to say, how many different kinds of such polynomials are there, where we are only interested in qualitative differences? The answer is simple enough: there is only one kind. All monic quadratic polynomials have graphs which are qualitatively similar to the graph of $y=x^2$. (In fact even more is true: See 1.7). This qualitative type can be described as follows: concave-up with slope increasing from $-\infty$ to ∞ (i.e. unboundly large negative to unboundly large positive.)

The situation for degree 3 is more interesting. Can you give a qualitative classification of monic cubic polynomials? It turns out that there are three essentially different kinds represented by the three particular graphs at the right.



The graph of every monic cubic polynomial belongs to one of the above three types.

Precisely what is it which distinguishes these curves qualitatively? When you have answered this question you should be in a position to prove that every monic cubic polynomial is similar to one of these 3 types. In fact can you provide a simple rule which, given a particular function $f(x)=x^3+ax^2+bx+c$, will enable me to quickly tell whether it is of type I, II, or III?

Now, when you have done this, you might try attempt a similar classification of monic quartic polynomials. How many types? How do you tell the type of a given quartic?

11.4 I am interested in the way in which the largest root of the polynomial $x^3-x+\alpha$ depends on the parameter α.

(a) If we let r denote this largest root, then r is a function of α. Give me a qualitative description of this function. Draw a rough sketch of its graph.

(b) Can you give me any quantitative information about r. For example can you calculate $dr/d\alpha$?

11.5 A point P is a point of symmetry for a curve if for every straight line through P, the set of points of intersection of the curve and the line is symmetric about P. Thus a straight line is symmetric about any point on it, and an ellipse is symmetric about its centre. From looking at the graphs of cubic curves, you might suspect that every cubic curve is symmetric about its inflection point. Is this true? If so prove it. [Hint. Make life computationally easy for yourself! For example, by translating the curve, you can assume that the inflection point is at the origin. What does the general cubic polynomial with inflection point at the origin look like algebraically!]

11.6 (a) If a polynomial $p(x)$ contains only even powers of x, then its graph is symmetric about the y-axis. Interpret this symmetry property algebraically and prove this simple result. Is the converse true? Can you prove it? If you can't prove the converse in general, try it for a particular class of polynomials, say, all polynomials of degree 4.

(b) Using the results of (a) find necessary and sufficient conditions on the coefficients a,b,c,d and e for the polynomial $ax^4+bx^3+cx^2+dx+e$ to be symmetric about the line $x=1$.

11.7 What can you say, qualitatively, about a function $f(x)$ which for some constant k satisfies the quation $f'(x)=kf(x)$ for every x. Draw pictures to illustrate the different possibilities for f.

CHAPTER 4.

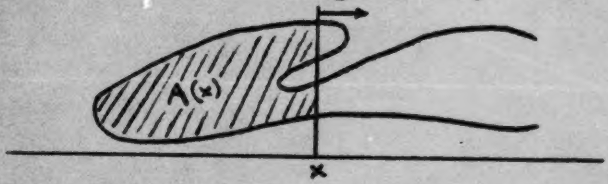
The Fundamental Theorem of Calculus.

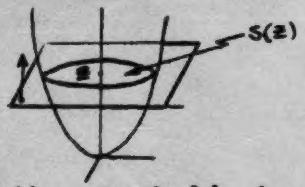
The idea behind the FTOC is powerful and pervades much of Mathematics, but is rarely appreciated or even apprehended by students who are usually taught a formula which they find difficulty in remembering. The difficulty is often compounded by presenting the students with two formulae, a first form and a second form, both of which apparently must be memorized.

The idea in its most basic form is the following. We are anxious to evaluate a quantity α . We recognize that α is, in a natural way, the value of a certain function $A(x)$ at some point $x = b$. The function $A(x)$ is no easier to write down than the number α , but the derivative $A'(x)$ is, for one reason or another, possible to find. We then try to get $A(x)$ from $A'(x)$ by antidifferentiating. If we can, we then evaluate $\alpha = A(b)$. In this form the idea applies to problems in physics, economics and biology to name just three areas.

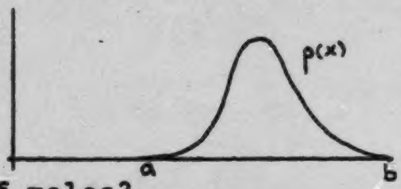
In the standard FTOC the quantity α to be determined is the area under a graph $y = f(x)$ above an interval $[a,b]$. The function $A(x)$ is taken to be the area above $[a,x]$ and $A'(x)$ is seen by a simple geometrical argument (which need not at this stage be done rigoriously but it can be easily seen to require something like the continuity of f) to be $f(x)$. If we can antidifferentiate f , we can find $\alpha = A(b)$. The idea is simple and geometrically compelling. Once grasped it will never be forgotten.

The underlying geometric idea that the derivative of A is f should be discussed somewhat more generally. If we have any planar region and we generate area by moving a straight line parallel to itself at unit speed then the rate $A'(x)$ at which area increases is the width $w(x)$ of the cross-section at x. It is then a simple matter to up the dimension by one, and notice that a volume generated by a moving plane increases at a rate equal to the area $s(z)$ of the cross-section at z. Armed with this idea it is a simple matter to calculate the volume of the parabolic bowl $\{(x,y,z) | x^2+y^2 \leq z \leq 1\}$. This is a problem which students usually find almost impossible to do a year after they have "learned how". They recall something about disks or cylinders of width dz but can't remember how to write the integral.





Let me give one other example of the idea.



PROBLEM. Suppose $p(x)$ is the probability density function for heights of adult males in North America. What is the average height of this collection of males?

SOLUTION. Let me first emphasize that the student does not need a comprehensive course in probabilities and expectations to understand the problem. In fact such a course is possibly the worst preparation. Everything can be carefully explained so the student has a good feeling for just what the average height should be. His problem is to find the correct mathematical expression.

Let $H(x)$ be the average height of the subpopulation of all males with heights between a and x. Let $A(x)$ be the area under the graph of p above $[a,x]$. We want $H(b)$. Let us find $H'(x)$. For any $h > 0$, $H(x+h)$ is a weighted mean of the average heights of men in $[a,x]$ and those in $[x,x+h]$. If h is small the first group has average height $H(x)$ and the second has average height approximately x. Thus

$$H(x+h) \cong \frac{A(x)H(x) + x\big(A(x+h)-A(x)\big)}{A(x+h)}$$

$$A(x+h)H(x+h) - A(x)H(x) \cong x(A(x+h) - A(x))$$

if we divide by h and take limits we get that $A(x)H(x)$ has derivative $xp(x)$. Then $H(b) = \int_a^b xp(x)dx$, since $A(b) = 1$, and $H(a) = 0$.

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Does Calculus Have a Gist?

The Chain Rules and the Fundamental Theorem

As an application of the "weakest link" principle, from the purely logical point of view every part of a proof is just as important as every other part, and every part of a logically connected structure is just as important as every other part. But, especially in the matters of pedagogy, understanding and exposition, the purely logical point of view is far from adequate. 42 years ago, H. Weyl wrote, in the Preface to his "The Classical Groups",

> *"The stringent precision attainable... has led many authors to a mode of writing which must give the reader an impression of being shut up in a brightly illuminated cell where every detail sticks out with the same dazzling clarity, but without relief. I prefer the open landscape".*

Weyl's concerns are still generally valid today, and nowhere more so than in the realm of introductions to Calculus, the mode of whose lengths appears to be approaching 1000 pages. The need for salient features is particularly great in works of this magnitude. Unfortunately, the greater the need the more they tend to get buried in the great mass of detail. Yet, who will cast the first stone, in view of current market demands for such works? And who will step forward and advise an author of a text of pyramidical (the Egyptian variety) magnitude that quite a few things should have been said, or emphasized, that weren't? Indeed, there may be an element of perversity in all this, as was revealed to me by a colleague in saying "I like these texts for my courses - they make me feel needed." Unfortunately, ideal lecturing conditions are becoming just as remote as the conditions needed for writing a good text. Fortunately, there may be a way out via articles in expository journals: at least we can try.

What characterizes good exposition is in part the extraction and presentation of a gist (or essence) of an argument or structure. This may stand on its own, or it may be a helpful preliminary to detailed logical analysis. For definition addicts, we may define a (note the indefinite article) gist as a part (preferably small), the knowledge of which makes it very much easier to understand the whole - in Mathematics a really good gist reduces the rest of the story to familiar routines (perhaps strenuous, but not difficult). The concept is clearly not at all precise, and is psychological rather than logical.

Our thesis here is that the overall structure of Calculus has a gist, consisting of the Chain Rules and the Fundamental Theorem, and that this gist has itself a gist, consisting (in a sense) of a simple algebraic identity. The details are all enshrined in the literature, but too often buried or implicit.

Our task is the modest, but, it is hoped, useful, one of shining a spotlight in order to counteract the deadly evenness of illumination to which Weyl referred.

More important still is the role of gists in the creative, rather than expository, process. Here they are indispensable, as they are all that is available when a result or topic is taking shape for the first time. For mathematicians this goes without saying (and hence isn't said) - but for their students it usually has to be pointed out.

Our main message is in I. In II we focus on the Fundamental Theorem by itself, connecting it with summation by differences. III is a variation of II, with emphasis on how one might be led to the Fundamental Theorem.

I

A significant part of the overall structure of Calculus is an exploitation of one trivial algebraic identity, which we write in the form

$$\Delta y = \frac{\Delta y}{\Delta x} \Delta x$$

(Here, Δy and Δx, and Δt to follow, have the interpretation of increments in the variables y, x and t. These are assumed to be related in the usual way: *the present article is written for teachers, who can fill in the details themselves.*)

1. If we divide both sides by Δt, and take the limit as the Δ's all approach 0, we get *The Chain Rule of Differential Calculus*,

$$D_t y = D_x y \cdot D_t x$$

(under certain conditions).

2. If we multiply both sides by $f(y)$, and take the limit of the usual sum (the details will be familiar to the reader) we get *The Chain Rule of Integral Calculus*,

$$\int_p^q f(y)dy = \int_{x_p}^{x_q} f(y) D_x y \, dx$$

3. From the Chain Rule of Integral Calculus, the *Fundamental Theorem* follows almost trivially.

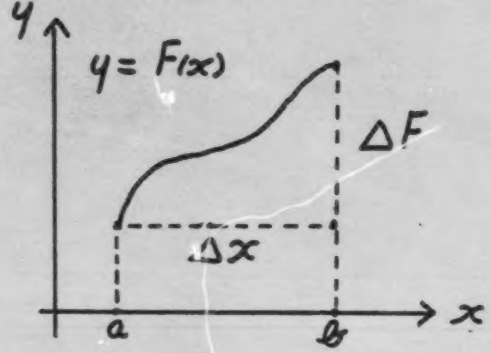
If $F' = f$, then $\int_p^q f(y)dy = \int_{y=p}^{y=q} 1 \, dF(y)$, by the Chain Rule as in 2, and the right side, from the definition of the integral, is at once seen to be $F(q) - F(p)$, as required.

It is of course true that many details need to be filled in in order to produce acceptable proofs of these three theorems.

But this does not adversely affect the expository value of such a treatment - in fact, it is in large measure a corollary to the "spotlighting" approach.

II

We continue with another "proof" of the Fundamental Theorem which is revealing in other ways. It involves a result which captures a significant part of the Fundamental Theorem, but which is so simple that its proof is, literally, trivial, even in complete form.

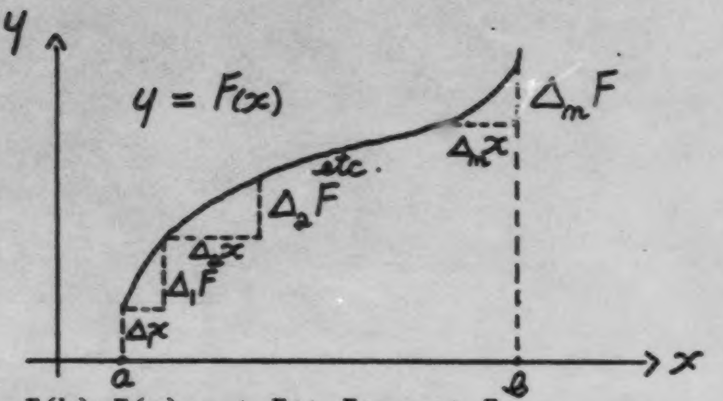


In the situation pictured,

$$\begin{aligned} F(b)-F(a) &= \Delta F \\ &= \frac{\Delta F}{\Delta x}\,\Delta x \\ &= [\text{average rate of change of } F(x)]\,\Delta x \end{aligned}$$

(so far: no sums or limits)

A slight elaboration looks more like the familiar theorem. We consider n steps instead of one.



$$F(b)-F(a) = \Delta_1 F+\Delta_2 F+\ldots+\Delta_n F$$

(note that this expresses the sum of a telescoping series)

$$= \frac{\Delta_1 F}{\Delta_1 x}\,\Delta_1 x + \frac{\Delta_2 F}{\Delta_2 x}\,\Delta_2 x+\ldots+\frac{\Delta_n F}{\Delta_n x}\,\Delta_n x$$

$$= \sum_{i=1}^{n} [i\text{th average rate of change of } F(x)]\,\Delta_i x$$

This result is exact and trivial (one is embarrassed to write it out), yet it very strongly suggests the Fundamental Theorem

$$F(b)-F(a) = \int_a^b F'(x)dx.$$

This approach to the Fundamental Theorem, which can be expanded to a complete proof, shows it to be the telescoping series (summation by differences) method of summing a series, with a limiting process (not at all trivial) thrown in. The "cute trick" part of it involves using the derivative to get the telescoping effect.

III

We conclude with an adaption of II which suggests how one might be led to the Fundamental Theorem

$\int_a^b f(x)dx$ is a limit of a sum. Let us ignore the limit and concentrate on the sum,

$$\sum_{i=1}^{n} f(x_i)\Delta_i x \quad \text{(with the usual meaning).}$$

The method of summation by differences suggests looking for an $F(x)$ such that $f(x_i)\Delta_i x = \Delta_i F$ for $i = 1,\ldots,n$. The sum will then be simply the total change in $F(x)$, namely $F(b)-F(a)$.

But the above equality amounts to

$$f(x_i) = \frac{\Delta_i F}{\Delta_i x} .$$

Ultimately, $\Delta_i x$ is to tend to 0. This all suggests that we try an $F(x)$ such that $f(x) = D_x F(x)$. Reversing direction, we are led to the Fundamental Theorem, and a proof of it along the lines of II.

A criticism of the point we have been making is sometimes given on the grounds that it is trivially true that every argument has a gist - one needs only to take a few trivial steps in the proof and call them a "gist". But our definition of a "gist" contains the phrase "the knowledge of which makes it very much easier to understand the whole". Judgment as to whether this requirement has been met is no doubt largely subjective, but this does not make it meaningless. In fact, it is this kind of judgment which one must come to grips with in the art of teaching mathematics.

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Applications of Mathematics

More Discrimination is Needed

Much recent discussion about mathematics education has to do with applications. Our thesis here is that the total range of "applications of mathematics" is so vast and varied in both substance and nature that it doesn't make sense to lump all applications together for the purposes of mathematics education - as if their mere applicationhood was sufficient to make them valuable.

There is a wide variety of qualities which applications of mathematics can possess, and a full appreciation of this variety is necessary if our assessment of the role of applications is not to be naive.

1. Being internal to mathematics. (e.g. algebra applied to geometry)
2. Meeting needs.
3. Being in demand. (not the same as 2.)
4. Increasing our technological power.
5. Contributing to our understanding of the world.
6. Being technically interesting in themselves.
7. Generating new mathematical concepts.
8. Providing intuitive interpretations of mathematical concepts and results.

9. Being good examples of the activity of mathematizing.
10. Bringing various areas of mathematics into interaction on a common problem.
11. Being essentially branches of mathematics themselves.

What matters is how we respond to these various qualities. Which are most important for the kind of mathematical education we have in mind? Do we equate applicationhood with having utilitarian value? (A fundamental mistake.) Does the pure-applied dichotomy correspond to the dichotomy of intellectual and academic - practical and real-world? (The same mistake.)

Attention to such issues should help, among other things to avoid the prevalent misuses of "applications": pseudo-applications masquerading as applications; applications pretending to be real motivations when they are not; applications emphasized in a way which is, without justification, condescending toward pure mathematics.

As a limiting case, and in order to avoid being misunderstood, we mention that of the "null application". Any discussion of the role of applications must acknowledge the fact that mathematics which is devoid of external applications is far from useless or trivial. Moreover, in the form of games or puzzles, some interest in this kind of mathematics is close to being universal. It has an important place in mathematics education.

A broader classification is the following. Applications of mathematics are studied for two reasons - utilitarian and intellectual (academic). In the latter case, applicationhood is not what is significant. Perhaps in this case one should refer, not to "an application of mathematics" but rather to "an application-generated mathematical theory". What one studies is then not the application, but the mathematical theory.

We conclude with a few brief comments on 1-8. 9, 10, 11 do not seem to call for any remarks.

1. Such applications should be sought out and pointed out more often than is generally done.
2. (Note that we are not identifying needs with mere desires or with what is marketable.) Such applications are, at least in the long run, related to the social function of mathematics. Their power to motivate is very great, but their role within a particular course of mathematics is less obvious.
3. Here there will be pressure from outside to include such applications in our teaching. Since time is limited, and so many applications are of greater educational value, being in demand, by itself, shouldn't count for much.

4. The value of technological progress, in itself, is of course debatable. But it usually comes with other values attached: the important thing is that these should be recognized.

5. There is a whole spectrum of ranking within this category. At the top are those applications which have to do with the most general (and hence most significant) truths - usually referred to as laws of nature. The downward direction is that of increasing particularity, or decreasing permanence. Such applications have a special status with respect to the aims of a liberal education.

6. An application can be of value solely on the grounds of providing a challenging problem with an interesting solution.

7. To the mathematician, this quality is the most important of all.

8. Here again the focus is on the mathematics. A classic example arises out of the flow of an incompressible fluid in a plane: this gives a powerful intuitive grasp of the notion of complex integration, and even of a number of important theorems.

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The following article, by Mario Lavoie, is a good example of how a particular application can combine many desirable qualities. Such examples of the setting-up of a differential equation are seldom attempted at the level Prof. Lavoie has aimed at: this is unfortunate. Timing is not our primary concern here, however.

The setting-up of this differential equation, beginning with a physical situation which should not be beyond the grasp of an intelligent 18-year-old who has had some common, everyday, experience with fluids, thick and thin , is a fine example of mathematization, is significant in that it involves a property (if not a basic law) of nature, has a strong component of geometrical intuition, can be used to give rise to the concepts of differentiation and integration, and itself has practical applications.

APPENDIX D

Objectifs de l'enseignement du calcul différentiel et intégral au collégial.

Avant de donner une application du calcul différentiel et intégral et d'en discuter les mérites, nous allons d'abord donner la problématique dans laquelle celle-ci va s'insérer. A priori, nous nous limitons à l'enseignement du calcul au niveau collégial. C'est-à-dire au niveau des Cégep pour le Québec et à la dernière année de "high school" et à la première année d'université pour les autres provinces. A ce niveau, trois objectifs fondamentaux militent en faveur de l'existence de cours dans cette discipline. Le premier est: (*i*) l'importance culturelle que cette discipline a prise dans notre société. En effet, point n'est besoin d'une recherche exhaustive pour se rendre compte que cette discipline est omniprésente dans les sciences physique et naturelle. De plus, le XXième siècle a vu les sciences humaines découvrir et utiliser de plus en plus cette discipline. Ainsi considère-t-on aujourd'hui comme presqu'indispensable de comprendre cette super technique. Ceci nous introduit au deuxième objectif. Il s'agit (*ii*) de l'immense potentiellité de transferts de cette discipline à d'autres sphères d'activités humaines. Mais pour que ceci puisse se réaliser il faut pousser un peu plus loin la connaissance interne de la discipline et déboucher sur (*iii*) l'analyse des techniques propres du calcul différentiel et intégral.

Dans ce contexte, l'application que le physiologiste français Jean-Louis Poiseuille a publié vers 1846 est très éclairante. Attardons-nous y quelque peu. Mais avant, il est bon d'exprimer clairement que pour nous, cette application n'a pas un caractère de nécessité absolue dans un cours de calcul différentiel et intégral, mais qu'elle représente un archétype d'applications qui se doit d'être explorée dans un tel cours, de même que d'autres.

Application à la dynamique des fluides: Vitesse d'écoulement d'un fluide visqueux dans un tube

Lorsqu'un fluide s'écoule dans un tube ou un autre récipient, c'est qu'une force agit sur ce fluide. Essayons de cerner d'un peu plus près la force nécessaire pour déstabiliser ce fluide. Disons d'abord, qu'elle doit surmonter la force de résistance due à la "viscosité" du fluide. Comment ceci peut-il se modéliser mathématiquement? Un gaz ou un liquide circulant dans un tube ou dans un récipient quelconque, forme une mince pellicule adhérant à la paroi du récipient. La force requise pour produire l'écoulement du fluide est celle qu'il faut fournir pour faire glisser le fluide sur la couche stationnaire. De la même façon, on peut modéliser un écoulement continu uniforme par un empilement de couches de fluide qui glisse les unes sur les autres. C'est ce qu'on appelle un écoulement laminaire (voir figure 1 a). Dans ce contexte, la force produisant un écoulement du fluide est définie comme étant la force requise pour produire un glissement d'une couche par rapport à l'autre. Elle est donc fonction de l'endroit du fluide où elle s'applique. Intéressons-nous maintenant aux facteurs pouvant déterminer la force due à la viscosité. Dans le contexte qui nous intéresse, c'est-à-dire où il n'intervient pas de grande masse de fluide, les facteurs qui semblent les plus importants sont: (*i*) les aires des couches de fluide qui seront potentiellement en contact, (*ii*) la différence des vitesses de ces couches et (*iii*) la distance entre ces couches. Les deux premiers facteurs étant proportionnel à la force tandis que le dernier lui est inversement proportionnel, c'est-à-dire que la force due à la viscosité est proportionnel à $\frac{A(v(r+\Delta r) - v(r))}{(r + \Delta r - r)}$. Notons ν le facteur de proportionnalité, qui dépendra entre autres choses du fluide que l'on étudiera. On l'appellera le coefficient de viscosité ou simplement la viscosité du fluide. Comme notre intérêt est d'estimer la force nécessaire pour faire glisser une lame de fluide sur une autre ayant comme seule force contraire la force due à la viscosité, nous définirons la force d'écoulement au niveau r par la limite lorsque la distance tend vers zéro de moins ν fois l'expression précédente, c'est-à-dire la force de viscosité au niveau r mais en sens contraire;

$$f_e(r) = -f_v(r) = -\lim_{\Delta r \to 0} \frac{\nu A(v(r+\Delta r) - v(r))}{(r+\Delta r-r)} = -\nu A \frac{dv}{dr}(r).$$

Il est à remarquer que les expressions précédentes sont constantes par morceau et que les intervalles de constances sont de l'ordre de grandeur de un (1) angström(Å) (10^{-8} cm), ce qui est le diamètre d'une molécule de fluide et indique l'épaisseur de la couche de fluide.

Considérons maintenant un tube de longueur ℓ et de rayon R (voir figure 1 b).

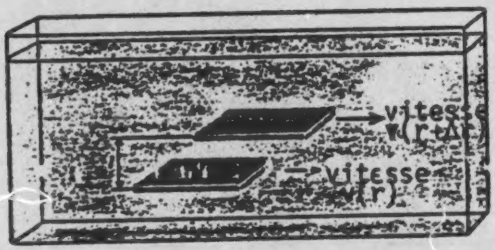

FIGURE 1 a)

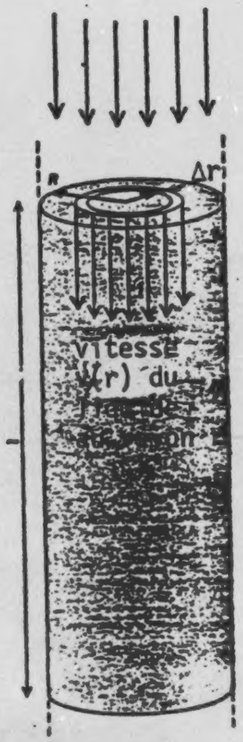

FIGURE 1 b)

Supposons qu'un fluide coule dans ce tube et que cet écoulement soit causé par une différence de pression P aux extrémités du tube. Ceci a pour conséquence de créer un écoulement ayant une vitesse uniforme, lorsqu'il est compensé par la force due à la viscosité. Comme on l'a vu précédemment, la viscosité du fluide entraîne la création d'une couche de fluide qui adhère au paroi du tube. Ainsi le fluide doit être poussé dans le tube par une force qui doit compenser la force de viscosité. Dans le cas d'une force due à une différence de pression et dans le cas d'un écoulement continue et uniforme, cette force pousse sur le cylindre central de rayon r, avec l'intensité $f_e = P(\Pi r^2)$. Celle-ci n'est compensée que par la force de viscosité située à la couche laminaire de rayon r qui, on l'a vu, vaut $f_v = \nu(2\Pi r\ell)\,\frac{dv}{dr}$. Cependant, à cause de notre modélisation, cette dernière force est constante par morceau. Comme ces morceaux sont relativement petits, on laisse varier continûment f_v dans l'équation $f_e = -f_v$, $r \in [0,R]$. Ceci donne que $\frac{dv}{dr}(r) = -\frac{P\,r}{2\nu\ell}$ pour $0 \le r \le R$ et par la technique de l'antidérivée on obtient $v(r) = \frac{P}{4\nu\ell}(R^2-r^2)$ pour $0 \le r \le R$, puisque $v(R) = 0$.

Il est à noter que la dernière hypothèse concernant la continuité de la force f_v peut facilement être enlevée, en utilisant un ordinateur, ou même une calculatrice de poche programmable.

Le profil d'écoulement correspondant à un écoulement laminaire est représenté par la figure 2 a). Cependant, pour fin de complétude, on

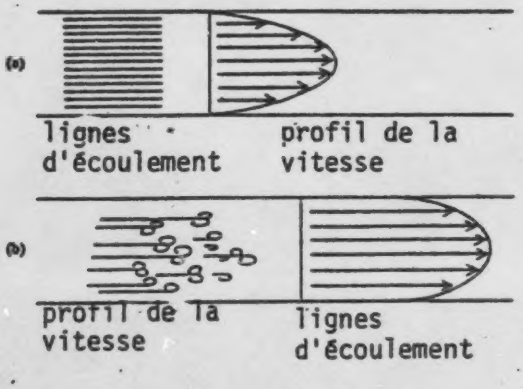

FIGURE 2

se doit de mentionner que ce ne sont pas tous les écoulements qui sont laminaires ou visqueux comme on les appelle souvent. On constate empériquement que les écoulements visqueux ne se produisent que pour des tubes de petits diamètres et de flux de faibles intensités. Les autres (voir la figure 2 b) produisent un écoulement que l'on dit turbulent et que notre modèle est impuissant à simuler.

Stratégies d'enseignement

L'application que l'on vient d'étudier offre plusieurs possibilités en ce qui concerne l'enseignement. D'abord il est incontestable que le résultat ait une valeur objective. Ceci est fondamental lorsqu'on enseigne à de jeunes adultes. De plus, cette application peut facilement déboucher sur des applications aussi concrètes que l'écoulement du sang dans les artères et les veines et le problème de l'hypertension (cf. Nathaniel A. Friedman, Calculus and mathematical models, Prindle, Weber & Schmidt, Boston, 1979, p. 331-333). Cependant, là ne s'arrête pas la valeur pédagogique de cette application. Si on y regarde de plus près, on s'aperçoit que l'on peut utiliser cette application pour introduire la notion de dérivée. En effet, en utilisant l'approche "par résolution de problème" qui consiste à faire dégager les idées fondamentales à partir de situations concrètes et de l'analyse de celles-ci, on guide l'étudiant vers la notion de gradient "moyen", puis presque par la force des choses, vers la notion de gradient "local". En répétant ce processus avec d'autres applications ou problèmes on peut dégager la notion de dérivée. De plus, s'ajoute à cela le fait que l'équation différentielle qui simule la réalité objective est particulièrement facile à résoudre et peut donc servir lors de l'introduction des primitives. Disons pour terminer la liste des avantages de cette application, que lorsque ce résultat est acquis on peut l'utiliser pour simuler le flux d'un fluide visqueux dans un tube, produit par une différence de pression aux extrémités du tube et obtenir ainsi une application de l'intégrale (cf. Nathaniel A. Friedman, p. 331).

Si on change notre point de vue pédagogique comme on l'a fait précédemment, on obtient alors une nouvelle façon d'introduire l'intégrale.

Il est à remarquer que ce type d'applications ne couvre pas toutes les facettes de la dérivée ou de l'intégrale. Par exemple, pour la dérivée la facette de la meilleure approximation linéaire n'est absolument pas couverte, tandis que pour l'intégrale c'est l'aire sous la courbe qui n'est pas touchée. Il faut donc faire bien attention lorsqu'on choisi ses applications ou problèmes pour que chacune des facettes des grandes idées y soient représentées.

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APPENDIX E

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CMESG/GCEDM 1980
Working Group B

APPLICATIONS OF MATHEMATICS FOR HIGH SCHOOL STUDENTS

Leader: Eric Muller

The group had available recent references on the subject and a number of articles, bibliographies and written comments were circulated. For a listing see the Appendix. Possible directions of study were explored and the group decided to devote its time to two main activities:

(a) The isolation of thos properties which are essential for a "good" application.

(b) The actual posing and solving of a number of applications. In this way the group hoped to test, in a limited way, the properties isolated in (a) and also hoped to generate new properties.

By the end of the three days of activities the group had agreed that the following properties were of importance:

(i) Better applications involve and challenge the student at his level of mathematical maturity.

(ii) Better applications are such that the situation can be understood quickly thereby allowing for a fairly rapid formulation of a problem whose solution is not immediately obvious, i.e. challenges the student as specified by (i).

(iii) Better applications have an element of surprise: "Oh, I didn't expect that!" - even though the student may have not expected anything else.

(iv) Good applications lend themselves to "what if" questions, i.e. more general or more specific problems which the student can raise.

(v) Good applications involve the student in an activity which is not purely mathematical, i.e. the application should not deal purely with abstraction, the application should allow for use of materials.

(vi) Good applications do not define a problem too specifically, i.e. they should allow for the sorting out of relevant information, should allow for different assumptions and different approaches both in the problem formulation and in the solution stages.

The activities of the group underlined the tremendous difficulties which arise when presenting an application, either in written form or in a teaching situation.

Of significance in a teaching situation, the application, even if it had all the ingredients mentioned above, becomes another exercise in mathematics if

(a) the application is taught as a mathematical problem (with a specific solution and drill in mind) i.e. the student is not allowed freedom of choice of interpretation or procedure;

(b) the class is expected to reach a certain stage of solution by such and such a time;

(c) interaction between students is not permitted;

(d) the teacher disallows any direction in which he/she is uncertain whether it will proceed to a solution or whether a solution exists at all;

(e) forbids related activities - cutting paper, rolling dice, using bicycles, jig saw, etc.

There appears to be a reluctance on the part of those who "teach" applications to describe them in writing. Once the application is frozen on paper teachers use them as another textbook problem with many of the failures listed above.

The group enjoyed three major activities:

(i) The design and analysis of a template to make right angle bends in circular heating pipes - an application identified by Mike Silbert, Hamilton Board of Education.

(ii) Loooking at the question "When you buy a 10 speed bike, are you really getting a 5 speed bike?" proposed by Hugh Allen.

(iii) Analyzing the game of coin tosing on a regular square grid which one sees at carnivals. The aim was to determine when the game would be fair.

Participants in Working Group B

Hugh Allen
André Boileau
Marin Hoffman
Saroop Kaul
Eric Muller
Tom O'Shea
Steve Whitney

APPENDIX

Recommended reading for the group:

1. "Applications: Why, Which, How" by Richard Lesh in Applications in School Mathematics. NCTM, 1979 Yearbook

2. "The Process of Applying Mathematics" by H.O. Pollak in A Sourcebook of Applications of School Mathematics. Joint MAA and NTCM publication, 1980

3. "A Classroom Teacher Looks at Applications" by Pamela Ames in A Sourcebook of Applications of School Mathematics. Joint MAA and NTCM publication, 1980

4. Bibliography from the NCTM 1979 Yearbook

5. Bibliography from MAA and NCTM publication, 1980

References

1. Applications in School Mathematics. 1979 NTCM Yearbook

2. A Sourcebook of Applications of School Mathematics. Joint MAA and NTCM publication

3. "Applications of Mathematics: Development Project" The Board of Education for the City of Hamilton

    (a) Set 1 for Intermediate Grades
    (b) Set 2 for Senior Grades

4. F. Mosteller, W. Kruskal, R. Link, R. Pieters and G. Rising. Statistics by Example, 4-volume series

5. Math Sciences: Roundtable, 4-volumes

6. The UMAP Journal, Vol.1

7. Michael Olinick. An Introduction to Mathematical Models in Models in the Social and Life Sciences

A number of sessions at the Fourth International Congress on Mathematics Education (Berkeley August 10-16, 1980) addressed the topic of Applications. At one of these sessions, Professor Max S. Bell, University of Chicago, circulated a listing entitled "A Preliminary Survey of Materials available world wide for the teaching of applications at the school level".

CMESG/GCEDM 1980
Working Group C

GEOMETRY IN THE ELEMENTARY AND JUNIOR HIGH CURRICULUM

Report edited by D. Lunkenbein and A. Boswall

1. Introductory remarks-approach to the discussions

Although there seems to be unanimity among the people concerned that geometry should be an essential part of the elementary and junior high school curriculum, it seems to be quite impossible to find a similar consensus about the particular contents and the various ways of teaching geometry at these levels.

In the past the teaching of geometry at these levels has been restricted to some exploratory activities with more or less particular geometrical shapes in preparation for measurement activities like length, area and volume, thus missing out on what many mathematics educators or curriculum experts of today would think of as essentially geometrical activities for younger children. What are these essentially geometrical activities? Are we able to describe them - at least by presenting some characteristic examples?

Some recent statements concerning the goals of education in this field[1] indicate rather clearly a broadening of the content and philosophy of geometry teaching in elementary and junior high schools, but they still leave us with what one could call two major problems of geometry teaching at these levels:

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[1] v.g. Activités géométriques à l'élémentaire, Fascicule F du guide pédagogique en mathématique pour l'élémentaire, 5e version, sept. 1976, p. 8 ff, Gourvernement du Québec, Ministère de l'Education, Direction générale du développement pédagogique.

(i) an absence of well defined, generally accepted subject areas; there is a lack of consensus about curriculum content in terms of desirable geometrical activities and in terms of the goals of geometry teaching;

(ii) an absence of structuring guidelines for geometry curricula; what are possible structuring criteria specific to geometry curricula (strategies, goals, skills and tools to be developed)?

In such a framework of questions and problems, complemented by resource readings of two papers by David S. Fielker[1] and by Isaak Wirszup[2], the work of the group was oriented towards two levels of considerations:

(i) discussion (description, analysis, criticism, etc.) of proposals for specific concrete geometrical activities;

(ii) more general discussion related to three major areas of concern to the participants of the group:

- What is a "good geometrical activity"?
- What are possible goals, tools or skills to be developed in geometry teaching?
- How can more geometry be introduced into the classroom?

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1. David S. Fielker, Strategies for Teaching Geometry to Younger Children, Educational Studies in Mathematics, Vol 10, No. 1, Feb. 1979.

2. Isaak Wirszup, Breakthroughs in the Psychology of Learning and Teaching Geometry, in Larry Martin, editor, Space and Geometry, ERIC/SMEAC Center for Science, Mathematics and Environmental Education, Columbus, Ohio, 1976.

2. Concrete geometrical activities

Two concrete geometrical activities were explored, investigated, analysed and compared. They generated and supported most of the more general contributions, so that it would seem important to report, at least briefly, on their nature and on the discussions they stimulated directly.

2.1 Fitting together equilateral triangles

Starting out with a collection of commercially available[1] equilateral triangles made of plastic, which can be hinged together along their sides, the initial task was to explore the nature and variety of geometrical shapes that can be created with this material. Among the suggested areas of exploration or investigation were two or three dimensional shapes, polyhedra of various sizes and shapes of faces, polyhedra whose faces are all congruent equilateral triangles (deltahedra), open and closed shapes and their stability, etc. The large number of different directions which investigations could take from such an initial exploratory situation was appreciated as an important characteristic of this activity. An investigation of convex deltahedra[2] was then pursued in more detail in order to indicate some possibilities of organising and systematising initial explorations, to show the gradual recognition of discriminating and unifying properties of such geometrical objects, and to appreciate the mathematical and geometrical character of such an activity.

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[1] TRI LOGIC, Mag Nif makes it, Mag Nif Inc., Mentor, Ohio 44060.

[2] An example of such an investigation is described in D. Lunkenbein, Groupings in the process of concept formation, to be published in FLM (For the Learning of Mathematics), 1980/81.

This activity was then discussed and described as one that immediately stimulated actual physical constructions, which, in turn, generated the realisation of several possible avenues for more directed investigations. The pursuit of any one of these directions generated an organisation and systematisation of concrete and mental actions, leading to the emergence of concepts and their development, thereby initiating a process of mathematisation. Geometrical objects, being first created physically, then more and more visually, were acted upon physically and mentally to explore variations and regularities in their properties. In this way concepts were clarified or enlarged, and spatial visualisation was forced, particularly when communicating the results of the investigations. Furthermore it was observed that the activities stimulated or the notions developed were not exclusively of a geometrical character. For example, geometrical and arithmetical activities or notions may be intimately interrelated. In order to count systematically the faces of certain polyhedra, one has to conceive spatial relations between the faces and, on the other hand, the comparison of various aspects of polyhedra may lead to the realisation of number patterns.

2.2 Exploring polyhedral shapes

A collection of wooden blocks representing various polyhedral shapes[1] was given to the participants in several exemplars. In order to contrast this situation with the preceding one, no suggestions for particular activities were made, so that the reflections of the group were directed toward possible uses of such a material for geometry teaching and learning.

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[1] D. Lunkenbein, Polyèdres - Ensembles de 26 blocs polyèdres, Dépt. de mathématiques et d'informatique, Université de Sherbrooke, Sherbrooke, P.Q. - This material and some of its intentions are described in D.Lunkenbein, H.Allard, C.Goupille, P.Allard, Polèdres - Rapport de recherche, Rapport no 26, Dépt. de mathématiques et d'informatique, Université de Sherbrooke, Sherbrooke, P.Q.

Contrary to the preceding activity, which stimulated reactions of a constructive mode, this situation provoked reactions of a more observational nature. These objects primarily appealed to an aesthetical sense, whereas the preceding situation stimulated interesting activities. The variety of given shapes, confusing initially, generated comparison activities and forced the observer to recognise, to see, to conceive qualities and properties. The approach to "noticing the geometry" seems to be observational, through looking at objects from a geometrical point of view, which in turn may be the origin for various manipulative activities: verification of qualities, classification, ordering, etc.. Activities like counting the number of faces, reconstructing blocks out of different materials, making or recognising drawings, etc. contribute in various ways to the ability to "see things" and to the development of spatial perception and of geometrical concepts[1].

The comparison of both activities brought out significant differences in their appeal to the learner, in the activities stimulated and the goals pursued, and, finally, in the ways they could be used as tools for teaching. In the first activity the learner feels much more in control of the situation, constructing and creating geometrical shapes and a geometrical environment by himself, the latter gradually unfolding in variety and complexity as he continues his investigation. In the second situation a complex geometrical environment is imposed on him, which he has to adapt to in order to control it. In the first activity, the learner takes the initiative and makes things happen, whereas in the second he is, initially, somewhat outside of the situation in front of him and has to observe what has happened before he makes things happen himself. So these activities call on and foster different abilities, thus appealing differently to the individual learner. Although allowing for a great variety of investigations, the first activity is very much determined by the availability of a particular material and restricted as to the geometrical content that can be derived. It primarily serves as an example of a creative mathematical activity directly stimulated by a concrete manipulative activity. The second situation, less determined by a particular material, may open a much broader conceptual domain to the learner but is more dependent upon teaching interventions to stimulate rewarding activities and conceptualisations.

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[1] Some research findings in this context are reported in D. Lunkenbein, Observations concerning the child's concept of space and its consequences for the teaching of geometry to younger children. Contribution to the mini-conference on "The Development of Children's Spatial Ideas", ICME IV, Berkeley, 1980.

3. Some general thoughts on geometry learning and teaching

In the following we attempt to sum up some of the more general ideas that emerged from the discussions of the preceding examples. They are grouped in three major areas of concern to the participants.

3.1 What is a "good geometrical activity"?

Since geometry is part of mathematics, a geometrical activity has to be considered as a mathematical activity. It is an activity of the individual which, in school, is to be stimulated, entertained and directed by a process of teaching. Whether or not such an activity is a "good" one, a rewarding one, depends very much on the individual. Hence, to describe or to develop criteria for a good mathematical (geometrical) activity becomes a rather subjective enterprise.

One has to reflect on personal experience in order to describe and evaluate the quality of one's relations with geometry or mathematics.

"Perhaps it is, for such an enterprise, as it is in life: if we hold someone dear, it is easy to tell of the qualities of the loved one; but if, conversely, we draw a list of marvelous qualities and then search for a person with these qualities, such a person might not be easily found. The discussion on geometry and on geometrical activities showed that we all have had some "love affair" with geometry; what we said, very often, was quite similar, but the experiences we referred to were very diversified.

Among the similarities of what we spoke of, three important elements were noticed:

> First, it seems that each one of these experiences offered some early success, if not immediate gratification of some kind. Otherwise we would have stopped and would not have invested a little of our energy.
>
> Second, such experiences contained a certain evolution: just like going along a road when the view suddenly opens on an attractive (conceptual) landscape giving us the assurance that nature is offering its richness and that we will spend a nice day here.
>
> Third, after having been caught by this landscape or in this "love affair" with a piece of geometry, the experience culminated in some kind of personal gain. The richness of the experience, its diversity, led to an insight which was felt to be essential. Having climbed a mountain we reached the top to discover the whole region in a single

> vision, discovering perhaps that our path toward the summit was not the best or the easiest one now that many paths were seen to converge there. We stood a while for joy and rest but, finally, left and went away because all adventures must come to an end and acquire the lightness of a memory.

So, for a geometrical activity to become a "good" one for some individual, it has to offer the learner the possibility of having his own "naive love affair" with geometry. In order to be able to judge an activity for such potential, the teacher must have had some appropriate experience. Then he will be able to choose, from his own experience, what he feels can quickly turn into a success, engage his students in a self-rewarding intellectual adventure and lead them toward those subjects enclosed in curricula that are believed to be of some importance". (Fernand Lemay)

Furthermore, a good geometrical activity, which has to be an acceptable challenge to the learner, seems to be stimulated by a situation, which demands action. Such action is generated by seeing how things are, and the things that might be done. Such original vision seems to be individual, so that the same situation may stimulate one person into action, while another stays inactive. Action, generated by vision, in turn makes such vision grow and increases the potential for further action. The growth of vision is an indicator of success in teaching.

Geometrical perception, as an important goal of the teaching of geometry, seems to develop from action. It can be considered to be an action itself, grown out of material dependency into a way of grasping something, which we can take with us all the time wherever we go. The development of geometrical perception is seen as an acquisition of personal power, which renders the student more competent in the use of himself. It contributes to the growth of the person, who can now enter into more dialogues with reality.

3.2 Goals in geometry teaching - some tools or skills to be developed

Intimately related to the preceding question is the specification of goals in geometry teaching or of tools or skills to be developed which would provide some criteria for judging the nature or the value of an activity in geometry. In a first attempt to explicate some of these criteria a list[1] of items was presented and discussed, ranging from general educational goals to specifically geometrical tools. In this list, goals and tools are not well differentiated, since a goal when reached may turn into a tool to be used and since the development of tools is to be considered an important goal of mathematics teaching. This list of 37 items by no means pretends to be complete or exhaustive. It is to be considered as a first attempt to provide an instrument that would help to develop and to appreciate "good" geometrical activities as particular mathematical activities in an overall process of education.

A. Educational goals/tools

1. Enjoyment
2. Excitement
3. Desire to explore
4. Confidence in a field
5. Desire to communicate

B. Mathematical tools also applicable to geometry

(general mental activities)

6. Pattern spotting
7. Seeing connections, associating

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[1] This list was originally drawn up by Alistair McIntosh.

8. Classifying
9. Recognising properties
10. Ordering
11. Decomposing into elements
12. Combining elements
13. Analysing
14. Transforming
15. Describing
16. Symbolising
17. Generating
18. Generalising
19. Hypothesis-making and -testing
20. Becoming convinced/convincing/proving

C. Mathematical tools with specific applications in geometry

21. Counting in geometry
22. Algebraic representations of geometry
    Geometric representations of algebra
23. Arithmetical representations of geometry
    Geometrical representations of arithmetic
24. Measuring (length, area, volume, angle, ...)
25. Diagrams/networks as specifically geometrical objects
26. Real life seen through geometrical perspective

D. Specifically geometrical tools

27. Names
28. Accurate drawing
29. Geometric construction on paper
30. Types of 2D and 3D structures
31. Properties of 2D and 3D structures
32. 2D representations of 3D objects
33. Constructing 3D objects from 2D representations
34. Visualising
35. Use of symmetry
36. Making, analysing, using tessellations
37. Concepts: vertical/horizontal, parallel/perpendicular, angle, perimeter, area, surface, volume, ...

The discussion of this list showed that such an instrument can be used to evaluate an activity to see whether or not it stimulates or develops some of the items mentioned; then such an activity would be worthwhile and would have its place in school. It also serves to indicate what, hopefully, is to be developed in geometry or mathematics teaching and could be a means to describe curricula. Further, the development of such a list regarding its completeness, its organisation, its interpretations and possible applications, may represent a rewarding task for future working groups on the learning and teaching of geometry in coming meetings of the CMESG.

3.3 How to get geometry into the classroom?

Geometry at the present time does not seem to be an important part of mathematics teaching. It plays a minor role in official curricula and an even smaller role in most classrooms. This situation seems to be in sharp contrast to the intrinsic worth attributed to geometry teaching, on the one hand, and to the potential of informal geometry as a valuable vehicle for arousing the interest and strengthening the confidence of both teachers and children, on the other hand.

Although such a situation is very much influenced by official curricula, it does not seem to be a very promising strategy for mathematics educators to try to change this situation exclusively by influencing curriculum development. The curricula we have, and the mechanisms for altering them, are more the product of social, political and administrative forces than the result of rational considerations of educational values. Furthermore, to legislate informal geometry into the curriculum is hardly a possibility, even if it were desirable.

Personal experience of geometrical activities, such as those described in 3.1, offeved to teachers through preservice or inservice courses accompanied by reflections on goals and on tools to be developed by geometry teaching, seems to be a more realistic, although rather lengthy approach, by which mathematics educators may provoke at least temporary changes. In order to make such changes more permanent, follow-up support to teachers should be provided. Such support could take the form of consultations, of workshops, of information on inexpensive resources and materials and of ideas for the integration of informal geometry with subjects other than mathematics, etc.. If such services are to be effective, efficient lines of communication will have to be invented so that information reaches the classroom teacher directly. In the case of the existence of examples of geometric activities in the classroom, where good teachers have set out with confidence to develop and follow an informal geometry strand, there is the possibility of "building outwards" from such classrooms to others, using the good classrooms as concrete examples of what can be done. -Furthermore it was suggested in the group that one way to increase the seriousness with which geometry will be taught in elementary schools in the near future is to demonstrate obvious and productive links between geometrical activities and number work. The perception that the one can illuminate the other might prove to be exactly the incentive needed, since all teachers (and parents) accept the importance of number work. Should small working teams be set up in various areas to examine this more closely?

More immediately still, those involved in preparing teachers and conducting inservice courses could, perhaps, make a "generous new gesture" of intellectual and professional support to those teachers who are ready to use some of their time in bringing the delights of geometry to their students.

4. Retrospection and outlook

No report on this particular working group would be complete without mentioning the positive and constructive atmosphere which reigned throughout nine hours of discussions. The written statements of some participants which are reproduced in the annex witness to such favorable ambiance. To some extent, this atmosphere was certainly due to the harmonious matching of the participants, which has to be considered contingent (accidental). But more important for the creation of such an atmosphere seems to be the spirit in which CMESG meetings are conceived as meetings where every participant is excepted to contribute and to feel responsible for the success or the failure of the working group. In this respect CMESG meetings differ considerably from customary conventions and stand the chance of being much more profitable to the participants. Such meetings can be expected to produce results which, through a team effort possibly to be extended over several meetings, will be beneficial to society in much more general terms. It is with such results in mind that we suggest that the work of this group be continued in subsequent meetings on topics or problems such as:

- Further development and organisation of a list of goals in geometry teaching and tools to be developed;
- Strategies and means to get geometry into the classroom;
- Research related to geometry learning and teaching;
- Preservice and inservice courses in geometry for teachers;
- etc..

It can only be hoped that the CMESG will continue to provide opportunities for working groups such as this one, which has been very profitable to all the participants and which might even have a lasting effect on the teaching of geometry in various teacher training institutions.

Following is the list of the participants of this working group:

Huguette Allard, Université de Sherbrooke
Alberta Boswall, chairperson, Concordia University
Cécile Goupille, Université de Sherbrooke
David Hawkins, University of Colorado
Frances Hawkins, Boulder, Colorado
Tom Kieren, University of Alberta
Fernand Lemay, Université Laval
Dieter Lunkenbein, chairperson, Université de Sherbrooke
Alistair McIntosh, Concordia University
Ronald Owston, University of New Brunswick
Arthur Powell, Educational Solutions Inc., New York
Ghislain Roy, Université Laval
Pauline Weinstein, University of British Columbia.

The group benefitted from the partial participation of Caleb Gattegno, Educational Solutions Inc., New York and Bernard Hodgson, Université Laval.

5. Annexes

5.1 Statements of participants

Whatever the age, the degree of culture, the amount of experience an educator might have, he can always, as long as he is capable of questionning himself without losing self-confidence, get back to the naïveness and the spontaneity of a child, when he is in front of an appropriate external stimulation (v.g. concrete educational geometric materials). From this new internal lighting, he may enter a mathematical adventure that will lead him to somewhere he has not been before.

That is what has impressed me a lot during these three days of workshop. In fact, the exploring period we got into every opening workshop produced this marvellous effect on one or the other member of the working group and his enthusiasm at that moment was a real and sane stimulation for everybody and acted as a sparkling pulse which has generated some deep and very engaged discussions on "why" and "how" to learn and to teach geometry today.

Huguette Allard

I want to express a very general reflection about the atmosphere in this workshop. The possibility exists that perhaps the presence of a poet, and a philosopher (to say nothing of a skilful king!) enabled the tillers of the soil to produce nourishing new varieties from simple seeds. We shall see.

Alberta Boswall

Neuf heures de réflexion et de discussion sur l'enseignement de la géométrie laissent des traces révélatrices d'un retour honnête du chercheur tant sur son acte d'enseigner que sur son désir d'améliorer de façon directe ou indirecte l'enseignement de la géométrie.

Lorsqu'on parle de formation, la dimension compétence demeure impérieuse voire même indispensable mais, et je m'en réjouis, les participants semblent unanimes à vouloir que la dimension humaine soit omniprésente et dans les objectifs à poursuivre et dans les stratégies à proposer pour un enseignement renouvelé de la géométrie chez les jeunes.

Cécile Goupille

In our first three-hour discussion we were unaware of missing our coffee break, like children who don't notice that it's recess time. I think our teacher then judged the conditions to be right. Happiness did not interrupt us, but gave support.

A first part was a study of deltahedra, and with these nice plastic triangles we seperately went in several directions - but then in time were ready to be guided toward a second phase by what David Wheeler calls "the directed intentions of a teacher", assumed that there were other starting points, and other directions, for another day. Indeed on the second day we were lowered into another part of the geometric forest, and this time no guidance was offered, no haste implied. The visual feast was enough.

Though our discussions ranged and sometimes perhaps rambled, there was enough - for such a group - of the logic of the concrete to keep us within a framework; the midcourse anxiety lest we ramble too much, and some careful summaries by members, were enough to bring back a sense of coherence and purpose. We had no straight and narrow path, but we must have evolved a pretty good sketch of a map; we didn't get lost.

David Hawkins

My general impressions of our working group:

1. It was the first time in my teaching career that educators and mathematicians worked together so well, There was no feeling of disrespect between the professionals. On the contrary, mutual respect and support was evident in each of our sessions. I very much appreciated the modesty of the members of the group.
2. With the help of David Hawkins I discovered a method of creating certain polyhedra from a cuboid by simply applying a few cuts - discovery in mathematics is, for me, the greatest joy.
3. I very much appreciated the two problem solving activities presented to the group. Active participation on the part of the members of a working group enhance the learning experience.
4. The excellent preparation on the part of the leaders of the working group promoted the very good feeling we experienced over the entire period.
5. The chairperson of the group was sensitive to the needs of the group as a whole and the individuals therein.
6. I appreciate having met each and every member of the group. I am richer for having made their acquaintance.

Pauline Weinstein

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CMESG/GCEDM 1980
Working Group D

THE DIAGNOSIS AND REMEDIATION OF COMMON MATHEMATICAL ERRORS

Group leaders: J.M. Sherrill, M. Bélanger

Reporter: S. Erlwanger

The group noted that although diagnosis and remediation is a common topic in mathematics education there was a lack of clarity regarding the courses, nature and extent of mathematical errors made by students. In addition there were questions about effective procedures for diagnosing and remediating learning difficulties. It appeared therefore that there was a need to develop a knowledge base in this area that would be of theoretical and practical value to both researchers and classroom teachers.

The approach proposed to the group was to survey present and future trends in diagnosis and remediation. The purpose of this report is to outline the areas discussed. Details of particular aspects are contained in the appendices.

The following trends were noted:

1. There has been increasing interest and activity in diagnosis and remediation during the 1970's. In particular, the creation of the Research Council for Diagnostic and Prescriptive Mathematics (RCDPM) would help to coordinate activities and improve communication among workers in the area.

2. A second trend has been the development of clinics at universities e.g. MEDIC at the University of British Columbia. Clinics have three functions:
   (a) service to the community. Students are referred by teachers to the clinics which provide diagnosis and remediation of learning problems.
   (b) training of graduate students as clinicians through courses on diagnosis and remediation and participation in the work of the clinic.
   (c) provision of research opportunities for faculty staff and students.

3. A third trend has been the emergence of three general approaches to diagnosis and remediation.
   (a) The most common approach adopted by mathematics educators and used in clinics involves one or more of the following procedures: the study of related variables such as reading, anxiety, attitudes, sex differences, etc.; the analysis and categorisation of errors; and the use of the task analysis model in the diagnostic-remediation cycle which consists of three stages: formal diagnosis using standardised instruments such as Key Math materials; informal diagnosis using locally developed instruments e.g. checklists and individual interviews; and remediation.

The group observed and discussed several examples of common errors made by students from examples provided by Sherrill. These examples showed particular areas in computation where errors frequently arise. However, a weakness noted here was that the causes of these errors was unknown and it was impossible to say how such errors could be prevented.

(b) A second approach used mainly with learning disabled, mentally retarded and mentally handicapped children is based upon an ability model. The emphasis here is on identifying the strengths and weaknesses of a student as a basis for providing specific remediation.

(c) A third approach indirectly concerned with diagnosis and remediation is the attempt to describe the 'child's mathematics'. This approach, described by M. Bélanger, required a careful analysis of interviews with children.

During discussions it was noted that these approaches served a different purpose and each had advantages and disadvantages. For instance, approach (a) provided a structured diagnosis-remediation cycle. In the short term it stressed remediating (curing the learning problem) but it did not in the long view assist the teacher to become aware of some of the causes of learning difficulties. Approach (c) however, could eventually make the teacher more aware of what some children learn that leads to problems. But the approach was not of immediate value to a teacher anxious to help a student.

The group also discussed problems that seem to affect developments in diagnosis and remediation. First, there was an absence of a coherent theory or theoretical framework, and research methods and instruments for guiding research efforts and classroom practise. There was also a lack of communication among workers in the area. It was anticipated that the Research Council for Diagnostic and Prescriptive Mathematics would in future improve communication and help to coordinate research and teacher-training efforts.

Secondly, it appeared that developments in diagnostic and remediation did not always take into account developments in the teaching of mathematics. In particular, it was noted that most of the work in diagnosis and remediation has so far focussed on paper and pencil computation at the intermediate level (grades 4-6). The following areas were discussed as possible directions for research in the future:

(a) Other student populations - primary, secondary and university. It was felt that very little appeared to be known of work at the secondary and university level.
(b) Other areas of mathematics e.g. geometry and algebra.
(c) The impact of technology in schools e.g. the use of calculators. Do calculators make paper and pencil algorithms redundant? How does this affect learning difficulties and errors children make? What kinds of learning difficulties and mathematical errors does the use of calculators introduce?
(d) A recurring theme during the discussions reflected the need for answers to the research problems in Sherrill's paper concerning the student, the teacher, teacher training, in diagnosis and remediation, methods and instruments for diagnosis and methods and materials for remediation.

Working Group D Report

Three appendices follow on subsequent pages:

Appendix A: State of the field: diagnosis and remediation

Appendix B: Diagnosis and remediation in methods courses

Appendix C: Interview with Jean

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APPENDIX A

STATE OF THE FIELD
DIAGNOSIS AND REMEDIATION

J.M. Sherrill

This paper was prepared specifically for the Canadian Mathematics Education Study Group (CMESG). The paper is organized as a three part (past, present, and future) presentation on the state of research in diagnosis and remediation in mathematics.

Part one, the past, is extremely brief. Part two, the present, will concern itself with a description of the mathematics education diagnostic and instructional centres (MEDICs) that are finally beginning to develop in North America as well as present an overview of research efforts. Part three, the future, will describe many of the areas in which research is needed in diagnosis and remediation in mathematics education. Finally, an extensive bibliography on diagnosis and remediation in mathematics education is presented.

PART ONE: THE PAST (0000 - 1966)

Though diagnosis and remediation have been part of the mathematics teacher's job since the beginning, very little specific organized research will be discussed in this paper. Brueckner (1929, 1930) reported the results of a research study concerning errors made with fractions in 1929 and a book on diagnosis and remedial teaching in arithmetic in 1930. Morton (1924) also dealing with fractions, reported the results of a study concerned with the analysis of pupil's errors. Buswell and Judd (1925) list 31 investigations which had as their explicit concern the diagnosis of arithmetical errors.

Much of the mathematics education research of the pre-1960 era was concerned with which mathematics to present, which algorithm is best for a particular operation, how should selected topics or algorithms be initially presented.

The remainder of this paper addresses itself with the "state-of-the-field" and what needs to be done in research in diagnosis and remediation in mathematics education.

THE PRESENT (1966 - 1980)

The year 1966 is picked to begin the present period because it was about that time that a workshop was held on the Kent State University campus that was to develop into a diagnosis and remediation clinic for mathematics education. Some of the research cited may pre-date 1966 since the studies may be more representative of the area being discussed. The Kent State University clinic turned out to be the first in a series of clinics that have been develcped in North America.

Clinics

The 10 - 15 centres that currently exist have many characteristics in common. For an excellent description of seven of the centres read, The Mathematics Clinic: Final Report, by Jerry Irons (ED 103 267).

All the clinics with which I am familiar (about 10) began by setting up a graduate course in diagnosis and remediation in elementary mathematics - specifically the intermediate grades. The concerns teachers have about diagnosis and remediation in mathematics is reflected in the courses becoming very popular. The courses eventually admit non-mathematics education graduate students, e. g., special education graduate students. The main prerequisite, however, is that the class members have had some teaching experience.

The course(s) is divided into two parts: 1) the theory and techniques of diagnosis and remediation and 2) the case study or studies. Some courses spend the first half on theoretical development of diagnosis and remediation, diagnostic techniques, formal and informal diagnosis, procedures for elementary mathematics remediation, and other topics to familiarize the student with the material s/he needs to know to do a case study and most of the second half working with public school students. The final part of the course is spent on compiling the results of the diagnosis and remediation, conferences, and final reports.

The MEDIC at the University of British Columbia spends the first third of the course in developing the theory and techniques of diagnosis and remediation and becoming familiar with the MEDIC checklists. Each class of the final two-thirds of the course is organized so that the first half is spent with the elementary school children and the second half with the rest of the diagnosis and remediation class.

In general, the clinics are established to serve three purposes: 1) Research, 2) Training of Teachers, and 3) Service to the community, though not necessarily in the given order.

SERVICE TO THE COMMUNITY: Teachers in the public schools, with the parents' permission, refer the student and parent(s) to a clinic. The parent(s) and student visit the clinic and set up a time for the diagnostic session. Quite often background material is available on the child. Students who have severe learning and/or emotional disabilities are referred to the Special Education Clinic; students who have reading disabilities are referred to the Reading Clinic.

Students who are given a diagnostic session usually go through two phases: formal and informal diagnosis. The formal phase may consist of the Key Math materials, the Slossen Intelligence Test, Stanford Diagnostic Achievement Test, and/or the SRA Math Diagnostic Kit. There are many other materials but the preceding four are by far the most commonly used diagnostic materials. The informal phase usually consists of a locally-developed checklist of mathematics objectives administered on a one-to-one basis, the MEDIC checklists are examples.

Once the diagnosis is completed a decision must be made --- will the remediation be done by the MEDIC? In many MEDICs once the child is accepted for diagnosis his problem(s) is also remediated. The clinics invariably charge a fee (e. g. $35 for diagnosis, $10/h for remediation). At the University of British Columbia's MEDIC no fee is charged. On the otherhand, not every child is remediated. Every child (excluding those referred to other clinics) is involved in a mathematics diagnostic session and the remediation package outlined, but the remediation is done by MEDIC only if the problem is one the faculty wants a particular clinician to study or if a clinician wants to work with the child.

The community is served by the clinic providing a detailed diagnosis of the child's mathematics difficulties and, in most cases, by the clinic providing the remedial service as well.

TRAINING OF TEACHERS: A synonym for clinician is teacher. The MEDIC clinicians are graduate students in education. In almost all cases the clinicians are teachers who are working on a graduate degree. The teachers receive invaluable training in both diagnostic and remediation techniques. The teachers then go back to their classrooms, become Learning Assistance Teachers, become university professors, of influence mathematics education in some other capacity. The assumption (and that is all it is currently) is that they will make use of the techniques they learn and the material they are given in whatever position they have upon completion of the course(s).

RESEARCH: There are very few areas of mathematics education that are more ripe for and in need of research effort than diagnosis and remediation. Some of the needed research will be discussed in Part Three of this paper. The present section is concerned with how the current clinic organization serves its purpose of Research.

Irons (1974) received funding to make a definitive study of the MEDICs in the U. S. A. and report his findings to his institution. Based upon his findings a MEDIC was started at his institution. The rest of the MEDICs discussed in this paper, however, developed in almost isolated conditions. Mathematics educators in different areas of North America seemed to have the same felt need and developed a MEDIC at their institution. Eight MEDICs are listed in Appendix A.

Fortunately, through reading the same reports (found in the ERIC system) and attending the same diagnostic and remediation sessions at conferences (mainly NCTM conferences) the faculties of the different clinics learned of the existence of other clinics. Communication among the clinics was and is still very difficult. To concentrate their research efforts and to avoid having each clinic re-invert the wheel, it was decided to hold a National Conference on Diagnostic and Prescriptive Mathematics. The conference was an instant success and the seventh annual conference was held recently in Vancouver, B. C.. Next year the conference will be in Hershey, Pennsylvania following the NCTM meeting in St. Louis, Missouri. Out of the conferences developed the Research Council for Diagnostic and Prescriptive Mathematics (RCDPM).

The preceding vignette was presented to show how much progress has been made in the last few years in organizing mathematics educators interested in diagnosis and remediation. One of the purposes of the organization is to concentrate the research efforts of the group in particular areas. One of the many difficulties with the current approach to research in mathematics remediation is that mathematics remediation is done, in most cases, on a one-to-one basis, therefore, the samples are very small. Having a network of clinicians attacking the same problem (as is done in the Soviet Union) may bring about a more realistic solution to some of the problems. Also the interchange of information, without the seemingly interminable delay of the publication process, helps all the researchers to stay "on track" of a particular problem as well as give direction to researchers working on other problems.

The individual clinic is also designed to implement research studies. It is the individual clinic that must carry out any research effort that is embarked upon. The individual clinic has as its clinicians graduate students trained to do educational research and interested in diagnosis and remediation in mathematics education. Working with them are university personnel who are trained and experienced researchers. The individual clinic also has access to students, the ability to organize diagnostic sessions in any manner needed, and the ability to produce remediation packages to any specifications. Speaking for the University of British Columbia's MEDIC, the individual clinic also has relatively free access to students in classroom situations.

One of the most important events related to the increased research efforts in diagnosis and remediation in mathematics was the forming of the RCDPM. Whether the RCDPM is a cause or a symptom of the renewed effort in diagnosis and remediation in mathematics is open for debate. The RCDPM does, however, offer a gathering place and focal point for people in North America interested in diagnosis and remediation in mathematics. Through the RCDPM's annual conference, monographs, and newsletter up-to-date happenings in the area can be disseminated. It has been felt for many years that the NCTM was not meeting the needs of people interested and diagnosis and remediation. It should be noted, however, that after an initial uncomfortable co-existence the NCTM has been very supportive of the RCDPM.

The growing interest in diagnosis and remediation is also reflected by the phenomenal growth of the RCDPM. The recent conference in Vancouver was attended by almost 500 people.

"Review" of Research

It is an adaptation of the RCDPM plan for organizing research that will be used in the research section of this paper. The research studies will be organized under the headings: Error Analysis, Related Variables, and Suggestions to Teachers.

Error Analysis

My experience has been that those clinics and individual researchers involved in diagnosis and remediation in mathematics begin their research efforts by surveying, categorizing, and analyzing errors children make in their work in mathematics.

While many studies could be described I have opted for the one with which I am most familiar---a survey of arithmetic errors implemented by UBC's MEDIC.

In an effort to find out the kinds of mistakes made in whole number computation with the four basic operations, the relative frequency of different error patterns, as well as other information, data were collected on 5440 students in grades 5 - 8. The computation test consisted of 12 addition, 12 subtraction, 12 multiplication, and 8 division problems. The problems represented different levels of difficulty, e. g., the eight division problems ranged from problems with a zero in the quotient to single-digit divisors with no remainder. Since the concern was with error patterns the students were provided with an addition table and a multiplication table.

The results met some of our expectations, but there were some surprises. Results that were not a surprise were:

1) The test and subtests reliabilities were well grouped and relatively high, ranging from the low of 0.76 for the subtraction subtest to 0.88 for the total test.
2) The success rate across operations (addition, subtraction, multiplication, division) decreased for all grades. It is interesting that subtraction and multiplication yielded similar results in both grades 7 and 8.
3) Within each operation, the success rate increased as one ascended through grades.

Results that were a surprise are listed below in increasing order of "surprise":

1) Even when given an addition table and multiplication table, students made a higher number of basic fact mistakes than was expected.
2) Overall, the success rate was higher than expected. The success rate for addition was 94%, subtraction 87%, multiplication 84%, and division 67%.
3) Practically all students perceived themselves as being very good at whole number computation.

There were approximately 37 000 errors made on the computation test. All were categorized. There were 48 mutually exclusive categories. Of the 48 categories, 17 dealt with place value. Knowledge of basic facts and place value are two of the leading factors in the breakdown of computational ability.

One of the ongoing studies in the series implemented by UBC's MEDIC dealt with the low achiever. It is the low achiever in computation that shows up at the clinic. The study followed up a finding that these students perceive themselves to be good at computation. Many of the students were interviewed with respect to their method of working a computational problem, why they think their method is correct, and why they feel they are good at computation. The reasons they stated are very interesting and are listed below:

1) AUTHORITY---the teacher, the book, or fellow students support the method as being correct.

2) SUCCESS---the method works often enough to convince them they are correct. Several subjects were accurate about their success rate (ranging from 30% to 60%) and they interpreted the performance as being good.

3) FATALIST---the method is simply the way it must be done.

The UBC study involved a paper-and-pencil survey approach with the analysis performed after the fact. The technique of interviewing a student and actually observing the errors he makes is currently considered more acceptable. Though the interview technique was first reported by Uhl (1917) as an approach to diagnose errors it use was not that common until the present period. As more North American researchers adopt non-empirical techniques and put less emphasis on the quantity of data and more on the quality the interview technique will probably become even more common.

Again, many studies could be cited with each of their findings with respect to the number and types of errors children make. After giving a grand total for the four basic operations with whole numbers I will cite 3 studies representing important areas of error analysis not previously discussed.

In an excellent review of the literature on computational errors with whole numbers Burrows (1976) sums up that "The studies have revealed a total of 35 unique addition errors, 33 subtraction errors, 54 multiplication errors, and 71 division errors." (p. 9)

Gregory (1980) studied the problem of how data on computational errors are collected. In his study the data were collected using paper-and-pencil tests, one group using answer sheets with multiple-choice format. Gregory was interested in whether the errors that were made were mathematical or clerical. This study is important in the light of the increased instances of large scale assessments. The results?

> "At least half of the 1255 errors made on the first administration were clerical errors. Clerical errors may have been the source of an additional 478 errors made, but the analysis performed did not allow for classification of these errors."(p. 187)

The results are made even more interesting when one considers the sample ranged from grade 6 to Algebra I students.

The final two individual research studies to be cited do not deal with computational errors; they are both studies in geometry. Perhaps more studies in error analysis will be in non-computational areas or, at least, on computation with sets other than the set of whole numbers now that the area has so many studies in print.

Bright (1980) reports on three studies (Tishin, 1975; Khamutina, 1975; Krutetskii, 1976) performed in the Soviet Union which deal with the formation of geometry concepts in normal and mentally retarded students. Some of the conclusions reached were as follows:

> 1. Retarded pupils are successful in distinguishing geometric figures (two-dimensional) which can be defined by a single essential characteristic. They are not successful in distinguishing figures which can be defined only be several essential characteristics.
>
> 2. Modeling or selecting a geometric form (two- or three-dimensional) is significantly affected by the pupil's facility with language.
>
> 3. Success in providing justification of choices of models of geometric forms is dependent on whether the form is defined by only one or by several essential characteristics.
>
> 4. Geometric bodies (three-dimensional) are better understood than geometric figures.
>
> 5. Lack of understanding of the differences among concepts of length, area, and volume is revealed in the use of incorrect symbolism.
>
> 6. Computing areas and volumes is easier when the dimensions are given than when dimensions must be measured. Computing areas and volumes is easier when the dimensions are given in whole numbers than when the dimensions are given in mixed numbers.
>
> 7. In comparing spatial features, pupils often focus on inessential characteristics.
>
> 8. Objects of similar dimensions are more difficult to distinguish than objects of quite different dimensions.
>
> 9. Structuring a problem is difficult.
>
> 10. Transferring solution techniques from one situation to another is difficult. (Romberg, pp. 192-193)

The other geometry study dealt with the use of frames of reference (Dockweiler, 1980). The results of the study present two areas of disagreement with Piagetian research. Three tasks were involved: Bottle, Pendulum, and Faucet. In each task the frame of reference (bottle or pendulum frame) was tilted and the subject had to draw the water line or pendulum position.

The first area of disagreement should come as no surprise to those familiar with the "North American Problem" in Piagetian research. There have been many studies implemented in North America that show that students do not attain specified capabilities in the age range specified by Piagetians. Dockweiler's study is no exception.

> Not only did a majority of 5th graders fail to correctly complete the tasks, but none of the age levels showed a majority with an ability to perform the task dealing with horizontals. These results would seem to suggest that a large number of subjects may never be able to correctly use a frame of reference in a similar situation. (Romberg, p. 222)

The second area of disagreement

> ...lies in the ability of students to apply a rectangular coordinate system as compared with the horizontal and vertical frame of reference. These results suggest that these capabilities are developed differentially and do not develop simultaneously or sequentially as Piaget's results suggest. (Romberg, p. 222)

In the May, 1979 issue of the Journal for Research in Mathematics Education Hendrik Radatz offers a different approach to error categorization---an information-processing classification of errors. Since the causes of errors may cut across mathematics content topics, the identification may be more successful by "...examining the mechanism used in obtaining, processing, retaining, and reproducing the information contained in mathematical tasks." (Radatz, p. 164) To this end Radatz offers the following five error categories: Errors due to...

1. ...language difficulties
2. ...difficulties in obtaining spatial information
3. ...deficient mastery of prerequisite skills, facts, and concepts
4. ...incorrect associations or rigidity of thinking

5. ...the application of irrelevant rules ore strategies.

Radatz is arguing for more general categories so we may be able to identify and eliminate entire classes of errors. There is another faction of mathematics education researchers that are approaching diagnosis and remediation in a very content-specific manner. Is either method of attack correct???

Related Variables

There is a danger, already, that the search for variables related to student ability in mathematics will follow the same path as the search for variables related to teacher effectiveness. After a century of searching for teacher effectiveness variables with the seemingly countless variables studied, Begle (1979) concludes:

> Despite all our efforts we still have no way of deciding, in advance, which teachers will be effective and which will not. Nor do we know which training programs will turn out effective teachers and which ones will not. (p. 29)

Instead of presenting an exhaustive (exhausting) list of related variables and all the research involving each, I will present a few related variables, some representative research findings, and close this section with some studies that generate more variables.

The first two variables are sex and spatial ability. It has been believed that males are superior to females in both mathematics achievement and spatial ability (Fennema, 1974; Macoby and Jacklin, 1974). It has also been estaolished that spatial ability is positively related to high-level mathematical conceptualization (Smith, 1964). This male superiority was believed to start in early adolescence and continue into adulthood. It could be believed, therefore, that sex, as a "variable", would be related to a subject's low performance in mathematics. It could also be believed that it was low spatial ability associated with females that was the variable related to low mathematics performance among females.

Guay and McDaniel's (1977) findings supported the contention that "...high mathematics achievers have greater spatial ability than low mathematics achievers." (p. 214) The findings also supported the contention that "...males had greater high-level spatial ability than females." (p.215) The Guay and McDaniel's findings were consistent across grades. The most interesting part of the study was the school grades used were grade 2 to grade 7. The results were very consistent with the literature, but the pattern formed at a much earlier age than had been previously established.

Fennema and Sherman (1978) carried out a carefully conceived and well designed study to gather hard data to substantiate hypotheses or dispel popular myths with respect to sex-related differences in mathematics. Their summary is important:

> The strong conclusion reached by the authors after two years of intensive study of sex-related differences in mathematics achievement of students in grades 6 - 12 is that when relevant factors are controlled, sex-related differences in favor of males do not appear often, and when they do they are not large. (p. 201)

While the results show "...there are not universal sex-related differences in mathematics learning", they also show that as early as grade 6

> ...girls expressed less confidence than boys in their ability to do mathematics, and the subject was clearly sex-typed male, especially by the boys. In the high school years these differential attitudinal influences continued and were joined by a host of other negative attiudinal influences, such as girls' perception of mathematics as being less useful to them and girl's perception of less favorable attitudes on the part of their teachers and parents, especially the fathers. (p. 202)

One of the biggest surprises in the Fennema and Sherman study was that "...no significant sex-related differences were found in spatial visualization." (p. 195) If the result can be substantiated, it is certainly in direct conflict with prior research. The research effort in this area may be seen to shift its focus from spatial ability to anxiety (confidence, self-image, etc.) in the search for the causes of sex-related differences in mathematics achievement.

The third related variable is anxiety. Anxiety is a psychological construct with a wide range of theoretical definitions. It has been estimated (Speilberger, 1972) that over 5000 articles or books on anxiety were published between 1950 and 1970. Due, in part, to the difficulty of determining a widely accepted and practical definition of anxiety, mathematics educators have delineated the area and are studying mathematics anxiety as a situation specific form of anxiety.

Certain "facts" appear to be established. Hill and Sarason (1966) reported that high anxiety was inversely related to arithmetic comprehension and arithmetic problem solving scores.

In his review of anxiety and mathematics learning Sovchik (1978) reported

> The psychological literature also suggests that sex differences in reported anxiety are minimal in early elementary years but that girls tend to report more anxiety in later school years perhaps because boys are

> culturally conditioned to defend against admittance of anxiety. (p. 9)

While the results stated in the two previous paragraphs have been supported by other studies there is another side to the issue. Hellman (1976) reported that "for 138 4th graders and 150 8th graders reported anxiety was not related to mathematical performance."(p. 10)

There have been numerous studies (only a few will be cited) that have asked the basic question---What is a slow learner in mathematics? Such studies generate descriptions that, in actuality, list variables considered related to mathematics achievement.

One of the most interesting results was reported by the School Mathematics Study Group (SMSG) in Report #5 by Herriot (1967). In creating the sample for the proposed study of slow learners it was noticed by the researcher that most schools were selecing existing classes of low-achievers. Simply out of curiousity Herriot decided to find out the initial reasons why children had been placed in such classes. Below are the reasons she was given:

1. Below grade level in mathematics achievement
2. Inadequate reading level
3. Slow worker in mathematics
4. Inaccurate computation
5. Fearful of mathematics
6. Antagonistic toward school
7. Apathetic, indifferent toward learning
8. Recent transfer to school
9. Chronic absentee

Other variables associated with slow learning in mathematics resulting from research (only one citation is given with each) are Stanford-Binet IQ (Jerome, 1959), reading (Ross, 1964), socioeconomic level of home environment (Dunkley, 1965), attitude toward mathematics (Callahan, 1971), self-esteem (Robinson, 1973), reflective/impulsive behaviour (Schwebel and Schwebel, 1974), and the list goes on.

Many more studies are needed on the interrelatedness of the variables and the dominance of certain variables over others before we can make practical use of the knowledge we have.

Suggestions to Teachers

One of the main goals of research in diagnosis and remediation in mathematics is to provide suggestions to teachers. The research findings are important not only to teachers working with slow learners but to all mathematics teachers. Again, the entire area cannot be covered in depth, but those findings that are cited are used to give direction to those interested in the area.

1. The teacher must establish a feeling of mutual trust. (Erikson, 1959; Taba and Elkins, 1966) Among other things this involves learning "...when to talk and when to let the student talk." (Taba and Elkins, 1966, p. 275)

2. The slow learner must do things. "Sensorimotor involvement with real objects (and pictures of objects) is a crucial preliminary to effective verbalization and conceptualization by the slow learner." (Schultz, 1972, pp. 14 - 15)

3. Initial learning tasks must be geared to readiness; ongoing learning tasks must be consolidated before additional dependent tasks are introduced; successive tasks must be properly sequenced and paced. (Schultz, 1972)

4. Practice (drill?) is important in maintaining a skill. (Nutting and Pikaart, 1969)

5. Goals need to be clearly explained and short term. (Cronbach, 1967)

6. Numerous studies have been implemented in the area of mathematics content. Data have been gathered with respect to teaching basic facts (McKillip, 1974), place value (Smith, 1973), all four of the basic operations (Hazekamp, 1977; Sherrill, 1979; Hutchings, 1975; Kratzer and Willoughby, 1973; Uprichard and Collura, 1977), etc.

Taking place value as one content example one finds studies delving into many facets of the area. Several studies (Smith, 1969; Uprichard and Collura, 1977; Scrivens, 1978) have concluded that more emphasis is needed on the skills of decimal numeration for computational skills. Smith (1973) concluded that one of the difficulties in teaching place value is that the concept of hundreds is introduced too early.

Diedrich and Glennon (1970), among others, have shown that the teaching of decimal numeration should concentrate solely on decimal numeration, i. e., spend no time on non-decimal numeration systems.

Barr (1978) goes even farther and suggests methods for teaching decimal numeration. He used the following methods:

The students used physical objects and counted...

A: ...by ones to 36, e. g., then learned 36 is 3 tens and 6 ones. They never counted by more than ones.

B: ...to 10, grouped, counted 10 more, grouped, etc. They then learned to write the proper numeral. They never counted past 10.

C: ...sets of objects with more than 10 and learned the symbols. Later they grouped by tens and counted using tens. To count 36 objects, they grouped by tens then counted 10, 20, 30, 31, 32, 33, 34, 35, 36.

On the posttest (achievement and application) the scores showed C > A > B, but not at a statistically significant level. On the retention test (4 weeks after the posttest) the differences were statistically significant and in the same order, C > A > B.

So as not to leave the reader with the feeling that the teaching of place value is a settled issue, it should be noted that deFlandre (1975) and Fishell (1975) did not find that understanding place value necessarily led to change in computational skill.

While much work has been done in diagnosis and remediation in mathematics that can be of both immediate and long term importance to teachers there is much, much more that remains to be done. It is to the future that I now turn.

PART THREE: THE FUTURE (1980 -)

There is, of course, no end to the work to be done in diagnosis and remediation in mathematics. This section starts with some of the very large tasks that confront us and closes with specific questions that are of interest.

Organization

As alluded to earlier there is a definite need for better organization. The group effort that is being discussed among the clinics via the RCDPM can be a giant step forward. The current clinics can be of great assistance in creating new clinics throughout North America. If a network of clinics can be created and if inter-clinic communication can be improved, the effect upon the research production of each clinic could be phenomenal. It may be that a research dissemination newsletter (as opposed to a journal) could solve the communication problem.

It is still true that graduate students produce the overwhelming majority of research done in mathematics education. If this valuable resource group could be attuned to the research efforts in diagnosis and remediation in mathematics education, progress could be made much faster. In the same area of development, undergraduates in teacher education faculties and practicing teachers need to be trained in the techniques of diagnosis, remediation, and research.

Research Techniques

To me one of the biggest breakthroughs in diagnosis and remediation research in North America is the increasing acceptance of studies which have been designed to increase the quality, not the quantity, of the data. The interview techniques, the "think aloud" techniques, the protocol analysis techniques are all in the process of improvement. While each has its limitations I consider them far superior to large group paper-and-pencil studies for our area of interest.

Level

Currently most of the efforts in diagnosis and remediation in mathematics education is directed at the intermediate level (grades 4 - 6). Much more effort must be expended at the primary level (grades K - 3). Diagnosis and remediation at the primary level can take on the approach of preventive medicine instead of waiting for the need of a cure. The surface has not even by scratched at the secondary level!

The effort, so far, seems to be heavily concentrated in the area of whole number computation. There is the tendency to overlook the fact that, in British Columbia, at least, over half the elementary curriculum deals with non-whole number computation topics. The proportion is even more extreme at the secondary level.

While I agree that the priorities in the past should have been given to the intermediate level where the demand was acute, the efforts now must be spread to all levels. In working with the mathematical learning difficulties of intermediate level children one gets the sensation of fighting a series of grass fires. One stamps out one fire only to move to one of the many others, without trying to prevent the fires.

Theoretical Decisions

To avoid the "grass fires" approach certain theoretical decisions must be made to direct the research efforts and put the research on sound footing.

First, there are several models for diagnosis and remediation. The two most commonly used ones appear to be the ability training model and the task analysis model. The quote below is from an unpublished paper by A. E. Uprichard (1977):

> In the ability training model the purpose of diagnosis is to identify processes or abilities (such as visual or auditory discrimination) that are strong or weak in the learner in order to prescribe remediation. Here the clinician has two options, he could either teach to a learner's strengths or remediate his weaknesses. (p. 1)
>
> The task analysis model is more content or discipline oriented. In this model assessment of academic skill development and instruction are designed to move the child from where he is to where we desire him to be in the academic skills hierarchy. The emphasis is on component skills and their integration into complex terminal tasks rather than the processes that presumably underlie the development of specific tasks. (p. 2)

Both models have proponents and opponents. A majority (maybe even all) of the MEDICs have adopted the task analysis model. This may be due to the hierarchial characteristics of mathematics content, especially computation. We may be guilty of the same logic as the drunk looking for his keys under the streetlight. Both models, however, need to be subject to research scrutiny.

So many mathematics educators accept the task analysis model that developing support for it has been overlooked. Hierarchies need to be created and tested. One should try a simple experiment. Take a group of teachers or education majors and have them do a task analysis of one of the four basic operations. Look at the results and the point will be made.

Formal -vs- Informal Diagnosis

All of the clinics described in Irons' (1974) paper used both formal and informal diagnostic techniques. Both types, of course, have advantages and disadvantages.

Formal diagnosis in a group situation has the advantage of time. In the time it takes to test one student one can test an entire class, but what does one have? The tests are paper and pencil. One has the results of the student's work so one can tell, e. g., that the student does poorly in subtraction. One might even be able to tell that the student does poorly in subtraction with a zero in the minuend. It is very difficult, however, to determine how the student is working the problem.

Informal diagnosis usually consists of some sort of checklist such as UEC's MEDIC checklists. The checklists are locally developed to fit into the framework of a particular clinic. The checklists are excellent for pinpointing the errors of students and determining the processes by which the child is attempting to solve the problems. When a clinician has completed a diagnostic session he has a much better chance of knowing exactly what the child is doing and having a remedial approach in mind. The checklists, however, are very dependent upon the abilities of the clinicians. It is the clinician that decides which problems are asked of the student; it is the clinician that decides how many problems will be presented; it is the clinician that decides the sequence of the items. It is the clinician that sees the work of the student and decides to alter the diagnostic session based upon what the child has done. Unlike the formal diagnostic materials which are suppose to be "teacher-proof", the checklists are very dependent upon the person using it.

The checklist takes about the same length of time as the formal diagnostic test, but only one student at a time is being diagnosed. The UBC MEDIC checklists can each take as little as ten minutes or as long as the student can last! It is suggested that any individual session not last more than 45 minutes. In oppostion to the time criticism, one should keep in mind that with the checklist most of the work is appropriate for the child. In formal diagnostic tests if one child is perfect and another miserable in addition, they both work the same addition problems.

The major research criticism of checklists is that the data collected cannot be compared to anything. There are no norms for checklist data. Two clinicians using the same checklist cannot even compare data in the statistical sense. Some work must be undertaken to define the relative roles of formal and informal diagnosis materials.

In leaving this section it must be noted that mathematical diagnostic testing in the affective domain is just beginning.

Research Framework

This writer feels that the focus of diagnosis and remediation research in mathematics education would be clearer if it were done within an overall framework of the area. Questions such as, "What is the role of clinical research?", "Can research on remediation or mathematics learning difficulties stand independent of research on mathematics diagnosis?", "Can clinical intervention research be developed as a research method?", and others could be answered with respect to an overall research framework.

If a research framework can be developed, researchers in diagnosis and remediation in mathematics education may be able to tell much quicker the areas that are in the most urgent need of study. A logical timeline might be developed, i. e., Study 6 might not have a theoretical basis until Study 8 rejects or does not reject specified hypotheses. The framework may allow researchers to organize their thinking so that they may focus on the strengths as well as the weaknesses in the area.

For a more detailed discussion of this section see John Wilson's article, "An Epistemologically Based System for Classifying Research and the Role of Clinical Intervention Research Within that System", in _Proceedings of the Third National Conference on Remedial Mathematics_, pp. 1 - 24.

Other Specific Research Needs

In this section are 22 suggested research problems. Some are large, some are small---all are needed. Though it may not seem obvious the writer did attempt to organize this section.

1. Entering skills must be developed. For diagnosis and remediation to serve as preventive medicine one must know which skills a child entering _any specified grade_ should

have. In addition the teacher (and clinician) needs to know the skills a child should have (must have?) to enter work on a particular algorithm.

2. Work needs to begin in earnest at the secondary level. Again checklists need to be created and validated. To focus attention on priority items, desirable existing skills must be determined.

3. Checklists for non-computational skills need to be created and tested by use.

4. Content objectives for mathematics at each grade need to be created. Many survey studies are still needed to determine the learning difficulties associated with each objective.

5. When a person makes a computational mistake is there actually one thought pattern or many causing the mistake? Experience dictates there are many. Research needs to tell the clinician that for a given mistake there are X patterns that could cause it, each of the patterns occurs y% of the time, and remedial package Z has the best success rate for a specified pattern. This is the ultimate goal for clinicians.

6. Are certain error patterns caused by particular instructional techniques? Which error patterns can be remediated by specified instructional techniques?

7. Can certain error patterns be grouped for remediation?

8. Which learner variables affect specified error patterns (ability training model)? Which learner variables can we control? What is the intersection of the two sets?

9. Can reliable and valid instruments be developed for early identification of the mathematics anxious student?

10. The role of verbalization in the comprehension of mathematical concepts needs continuing study to be clarified.

11. What are the variables that determine the most "effective" diagnostic session?

12. What are the variables that determine the most "effective" remedial session?

13. Do teachers trained in the techniques of diagnosis and remediation use the techniques after the training is completed?

14. Do teachers trained in the techniques of diagnosis and remediation make "better" teachers?

15. What do we remediate? Is the error pattern a cause or a symptom?

16. How does mode of instruction interact with anxiety (self-image, attitude, etc.)?

17. Content hierarchies need to be logically created and empirically validated.

18. Follow-up studies need to determine if the effects of the remedial packages are maintained.

19. Can some remedial packages be developed into effective teaching procedures for the initial presentation of an algorithm to an entire class?

20. What is the most efficient size for a remedial group?

21. How long should children be allowed to use physical devices? At what point should the student be required to use symbols only?

22. I'll end the list by at least referring to the myriad of questions with respect to the presentation of content. Which content? When should the content be presented? How should the content be presented? etc.

Names and Addresses of Selected Diagnosis and Remediation Clinics

Dr. Robert Ashlock, Director
The Arithmetic Center
University of Maryland
College Park, Maryland

Dr. Jon Englehardt, Director
Mathematics Clinic
Arizona State University
Tempe, Arizona

Dr. James Heddens, Director
Mathematics Clinic
Kent State University
Kent, Ohio

Dr. Boyd Holtan, Director
Mathematics Clinic
West Virginia University
Morgantown, West Virginia

Dr. Michael Hynes, Director
Math Center for Diagnosis
and Remediation
Florida Technical University
Orlando, Florida

Dr. Jerry Irons, Director
Mathematics Clinic
S. F. Austin State University
Nacadoches, Texas

Dr. David Robitaille, Director
MEDIC
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Dr. Ed Uprichard, Director
Mathematics Clinic
University of South Florida
Tampa, Florida

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DIAGNOSIS AND REMEDIATION IN MATHEMATICS METHODS COURSES

No unit on diagnosis and remediation is required in the elementary mathematics methods course at the University of British Columbia (Education 371). There are several instructors, however, that do have such a unit in their section of the course.

There is an emphasis in all sections of Ed. 371 on 1) the use of manipulatives, 2) how to go from the concrete to the abstract, and 3) different approaches to the same topic. While these three areas are not, per se, part of a unit on diagnosis and remediation they are necessary. It is a fact that so much of the material a teacher needs for diagnosis and remediation deals with what it takes (in an individual professor's opinion) to be a good teacher. What is described below is the material included in a unit entitled, Diagnosis and Remediation.

I. ERROR ANALYSIS --- The actual work of K - 8 students are presented and discussed. The Ed. 371 students have one of their student teaching practica during the academic year. I try to time the diagnosis and remediation unit so that it straddles the practicum. Seeing the actual work of the students they will face in the classroom is a sobering experience. It also makes them sensitive to errors as they work with their students during the practicum.

II. THE SLOW LEARNER --- I make use of the 35th Yearbook of the NCTM to describe some of the characteristics of slow learners that have been identified from research. The discussion also includes those variables that do NOT necessarily relate to being a slow learner, but have been used in the past to classify a student as being a slow learner. Due to the students' courses I do not have to spend time discussing students with severe learning disabilities.

III. MEDIC CHECKLISTS --- The unit includes very little on group administered diagnostic tests. The concentration is on the checklists developed at the UBC clinic. Items III - V of the unit are presented after the practicum since we are forbidden to give anything to the students that smacks of an assignment during the practicum. The checklists reflect the current B. C. mathematics curriculum and is an aid to the Ed. 371 students in organizing the area they will teach.

IV. INTERVIEWS --- We discuss techniques for interviewing students only from the perspective of identifying weaknesses. The emphasis, however, is on developing a programme for remediating the weaknesses. The class also views a videotape of a MEDIC interview. Since I can no longer require them to interview a student during practicum, I'm trying to arrange an interview situation using the Kids on Kampus programme.

V. REMEDIATION --- The discussion centres on using alternate teaching strategies and materials. The emphasis is on developing lessons (that I would consider to be) beneficial for "normal classroom instruction". There is also discussion on arranging the classroom so small groups or individuals can have the teacher's attention.

To repeat myself, in the area of remediation, I find very little I would tell the preservice students that I wouldn't have told them in any methods course. In simplistic terms remedial teaching differs negligibly from good initial teaching. I agree with Claude Gaulin that no matter how well a class is taught some students are not going to get out of a lesson what the teacher expected. What is needed is to sensitize every teacher to the students, be sure that every teacher has mastered diagnostic techniques, and be sure that every teacher has a variety of approaches in their repertoire for each lesson they present.

Can it ever be done?

J.M. Sherrill

APPENDIX C: INTERVIEW WITH JEAN*

Jean is a nine year old boy, in the third grade of a school designated by the CECM within the project "Operation Renouveau". At the time of the interview with Jean he was in a regular classroom, but the teacher had told us that he should have been placed in a special education class because "he's very weak in mathematics". The teacher adds that Jean always has failing grades on exams given each six week period. In a discussion with the teacher we tried to get additional information concerning Jean, but she only keeps repeating that "he's weak in mathematics", that "he should be in a special education class" and "that he never does what he's supposed to do". We asked the teacher for her permission to conduct an interview with Jean, and she allows us to take him out of class. We ask Jean, "Do you do calculations in class?" He replies, "Yes, additions". (Note: In French children say "des plus" for addition, and "des moins" for subtraction problems.) "Can you do this problem?" We ask.

$$\begin{array}{r} 279 \\ 142 \\ \underline{398} \end{array} \qquad \text{(Problem \#1)}$$

He adds aloound very fast: "9 plus 2 plus 8 equals 18" (instead of 19). He writes the "8" at the lower right and a "1" in the second column above the "7". We speculate that perhaps the sum "18" might be due to the rapidity in his mental calculations.

$$\begin{array}{r} 1 \\ 279 \\ 142 \\ \underline{398} \\ 8 \end{array}$$

He continues the problem adding the ten's and hundred's and arrives at the answer 818.

$$\begin{array}{r} 21 \\ 279 \\ 142 \\ \underline{398} \\ 818 \end{array}$$

We present him with the numbers arranged in a different order and ask him to compute the sum:

$$\begin{array}{r} 398 \\ 279 \\ \underline{142} \end{array} \qquad \text{(Problem \#2)}$$

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*Extract from a paper by Elie H. Martin, "L'analyse des erreurs en mathématiques decelles chez des enfants (de 8 à 11 ans) en difficulté d'apprentissage: un nouveau sentier pour les orthoédagogues?"

This time he gets 809. Once again he adds very quickly and makes a mistake in the ten's column. At this point it is not clear if his incorrect answers are due largely to the rapidity of his mental calculations or due to other factors. Jean does not seem to have realized that the same three-digit numbers are in the two problems. For Jean perhaps he sees no contradiction (two different answers) simply because he has not noted that "they are the same problem - only the order is changed". (Note: In this type of interview it is important to speculate as one goes along probing. These on-the-spot speculations guide the formulation of new questions. A very tricky thing to do!)

At this point we thought that perhaps by presenting the same numbers in a third order Jean might realize "they are the same problem" and he would seize that something is wrong somewhere. It doesn't work out that way as we shall see.

We propose a third order to see if Jean can in fact note "contradictions" in his answers: "Can you do this one?"

$$\begin{array}{r} 142 \\ 279 \\ \underline{398} \end{array} \qquad \text{(Problem \#3)}$$

This time Jean arrives at the answer 819. For us, adults, it is clear that the three "problems" (in the school sense of that much abused word) should have a single and unique answer, but Jean does not seem at all concerned by three different answers. Has he noted that they are "the same problem"? We now push to see what Jean will say:

Interviewer: Jean, look here (in problem #1) there is a 279; in the second there is also a 279; also in the third problem. Look - there's also a 142 and a 398 in each of the problems. Don't you think there should be the same answer for each of the three problems?

Jean: Of course not! It's not the same thing. (C'est pas la même chose)

Int: Why isn't it the same thing?

Jean: Because the hard numbers aren't in the same place (Parce que les nombres durs ne sont pas à la même place)

(Note: Jean uses the expressions "nombres durs" and "les nombres pas durs". (Very literal translations of "hard numbers" and "not hard numbers" are used here to keep the use of the French "pas")

Int: Where are the hard numbers according to you?

Jean: That one and that one. He points to the 9 and the 8 in the units column in the second problem:

398
279 (Problem #2)
142

Int: And in this one? (Third problem)

Jean: The hard numbers are in the lower part

142
279 (Problem #3)
398

In the tens column Jean also says there are hard numbers; they are in the upper part in problem #2 and in the lower part in problem #3.

Int: In the first problem are there some hard numbers?

279
142 (Problem #1)
398

Jean: Of course not! The 9 and the 8 aren't together. There is a 2 there. (He points to the 2 in the units column)

Int: And in the tens column are there any hard numbers?

Jean: No! there is a 4. (He points to the 4 in the tens column "which separates the 7 and the 9")

In the early part of the interview we were led to speculate that perhaps Jean's main difficulty is due to the rapidity of his mental additions, hence the resulting in minor errors. But at this point there seems to be more than this at play here. Once again we are led to speculate on "what's in Jean's head" and we might attempt to formulate this as follows (at least for the moment):

> "There exist combinations of two digits whose sum is greater than ten. Jean calls these 'the hard numbers'. A combination whose sum is less than ten, Jean calls 'the not hard numbers' (les nombres pas durs). Hard numbers must be placed one above another in a column. If they are separated by a small number then they are 'not hard numbers'. The sums of numbers depends on the position of the 'hard numbers'. If the order is changed, the answers will be different".

Now this is one interpretation of what might be in "Jean's head". We attempt other questions to negate or confirm these speculations.

Int: When I was in school sometimes I would do additions by starting at the top and add going down. But sometimes I would start at the bottom and add upwards. (We demonstrate using the third problem;)

(Procedure A) $$\begin{array}{r} 142 \\ 279 \\ \underline{398} \end{array}\big\downarrow \qquad \begin{array}{r} 142 \\ 279 \\ \underline{398} \end{array}\big\uparrow$$ (Procedure B)

Int: If I did it once doing it this way (procedure A) and once doing it this way (procedure B) do you think I would get the same answer?

Jean: (Very long pause)...Er...Here (Procedure A) the hard numbers (9 and 8) are at the bottom. But here (Procedure B) if you start at the bottoms, you start with the hard numbers. (Long pause).... ... It's not the same thing. No...you have to do it this way (Procedure A).

Int: But suppose I did it this way (procedure B) would it give the same answer as doing it this way (procedure A)?

Jean: But it's not the same thing!! If you do it that way (procedure B) it's not the same thing.

Int: It might give another answer?

Jean: You have to do it to find out.

Int: Do you think we can know before doing it?

Jean: Well, of course not! (laughs)

Jean seems to have found the idea ridiculous. In his mathematics the position of numbers is important, and thus to say ahead of doing it if the two procedures will give the same answer seems absurd to him ("you have to do it to find out"). In the arithmetic of mathematicans the commutative law holds, but in Jean's mathematics:

$A + B + C \neq C + B + A \neq B + A + C$, etc. This is consistent with his categories of "hard numbers" and "not hard numbers". Problems #1, #2 and #3 are not the same in this system, and the appearance of "different answers" is also consistent.

Is Jean a "learning problem" or a "creative mathematician"?

CMESG/GCEDM 1980
Review Group I

CALCULATORS

Group leader: Claude Gaulin

INTRODUCTION

In 1979, during the CMESG/GCEDM conference, a working group discussed many aspects of the problem of the utilization of minicalculators in schools, including the following:

- uses of calculators in Canadian schools: the current situation
- existing materials about pedagogical uses of calculators: reports and recommendations, books and articles, teaching materials, research papers and monographs, bibliographies, etc.
- situations when it is appropriate / not appropriate to encourage the use of calculators in the classroom
- uses of calculators for pedagogical purposes, e.g. to foster the understanding or the dynamic application of concepts, properties, algorithms, etc. or to investigate patterns
- some particularly interesting problems to investigate with a handheld calculator
- possible impact of calculators in the future upon curricula, teaching methods and evalution in schools
- potential value of "pseudo-calculators" (electronic learning aids like DATAMAN) and of calculator games
- sensitization of teachers to uses of calculators in schools: objecives and means to achieve it.

Concerning the work done then, those interested are invited to read p.81ff in the Proceedings of the CSGME / GCEDM conference held in Kingston in June 1979.

The 1980 review group on calculators had two objectives: updating of the information compiled last year and discussion of new aspects of the problem of the utilization of minicalculators in schools. Four people accepted to contribute and stimulate the discussions by means of four short presentations:

Claude Gaulin: *The situation at the international level*
Shirley McNicoll: *The current situation in Canada*
Alistair McIntosh: *Extension of a project in England*
Roberta Mura: *Calculator accuracy*

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THE SITUATION AT THE INTERNATIONAL LEVEL

A. Updating of information compiled last year

The number of publications about the utilization of minicalculators in schools continues to grow at an incredible rate. That rate is unprecedented in mathematics education, for any other topic.

Besides the existing materials compiled by the working group on calculators last year, it is important to mention the following, which have appeared since that time:

> *Calculators: A Categorized Compilation of References*, by Marilyn N. Suydam ERIC / SMEAC June 1979
>
> *Investigations with Calculators: Abstracts and Critical Analyses of Research*, edited by Marilyn N. Suydam ERIC Calculator Information Center Jan. & June 1979
>
> *"The Impact of Electronic Calculators on Educational Performance"*, by Dennis Roberts. Review of Education Research, Spring 1980, Vol. 50, No. 1, pp. 71-98

B. Recent recommendations in U.S.A.

Reference has been made last year to the position statements of the National Council of Teachers of Mathematics about Calculators in the Classroom (Sept. 1974 & Sept. 1978), as well as to the recommendations contained in the NACOME report ("Overview and Analysis of School Mathematics Grades K-12", Conference Board of the Mathematical Sciences, 1975) concerning the utilization of calculators for teaching mathematics in schools.

In April 1980, the N.C.T.M. has released an important report:

> *AN AGENDA FOR ACTION Recommendations for School Mathematics of the 1980s*

which is likely to be very influential in mathematics education in coming years. Among the eight major recommendations, the third insists that

> *"Mathematics programs must take full advantage of the power of calculators and computers at all grade levels"*

and leads to the following recommended actions:

> *3.1 All students should have access to calculators and increasingly to computers throughout their school mathematics program*
>
> *3.2 The use of electronic tools such as calculators and computers should be integrated into the core mathematics curriculum*
>
> *3.3 Curriculum materials that integrate and require the use of the calculator and computer in diverse and imaginative ways should be developed and made available.*

C. Increasing concern for calculators at the international level

The use of calculators in mathematics education has been much discussed in several international conferences, since the panel discussion on "What may in the future Computers and Calculators mean in Mathematics Education?" which was very successful during ICME 3 at Karlsruhe in August 1976. For example, an international colloquium took place in Luxemburg in 1978 about "Calculators in School Teaching". During the 5th Interamerican conference on mathematical education, held in Campinas (Brasil), one of the major activities was a panel on calculators and computers in mathematics education.

It may be of interest to observe that six activities during ICME 4, to be held at Berkeley in August 1980, will deal exclusively with uses of calculators for mathematics teaching.

On the other hand, as part of the Second International Mathematics Study, information has been compiled about the situation in various countries with respect to the utilization of hand-held calculators in schools. As a result of that work, Marilyn N. Suydam (Ohio State University) has prepared an important document entitled

"Working Paper on Hand-Held Calculators in Schools"
(March 1980)

which contains reports from 16 countries (including one from Canada, written by Walter Szetela, U.B.C.), as well as a Synthesis of National Reports, which certainly constitutes the most up-to-date document available about the situation at the international level. That document should become public during ICME 4 next August.

D. Research activities with calculators

(A copy of the paper Synthesis of National Reports referred to above was handled to each participant in the review group for internal use. Comments were made about research activities with calculators, following the content of pages 74-75.)

See appendix: "Synthesis of National Reports".

Claude Gaulin

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THE CURRENT SITUATION IN CANADA (summary)

During a sabbatic leave from McGill University in 1978-79, I conducted a survey of Elementary School Mathematics in Canada, specifically: the nature of provincial curricula as perceived by ministries of Education, school and teacher education personnel. Based on the responses from 111 interviews held in all ten provinces over a ten month period, I indentified common priorities and concerns and reported these in a document dated January, 1980. Among the brief observations noted was the lack of recognition of the future impact of calculators on elementary mathematics.

In response to a question regarding the use of calculators, six Ministries of Education reported that no definite policy had yet been established (Nfld., P.E.I., N.B., Ont., Sask., and Alta.), two provinces reported a policy at secondary level only (Man., B.C.) and two provinces were considering usage at the elementary level (N.S., Que.).

When asked a similar question, school and teacher education personnel agreed that greater use of calculators should be considered, but only at the upper elementary levels. Among the school areas where there have been research studies conducted or where calculators are being used as a supplement to the regualr program (in computational skills), where the following:

- Montreal: Protestant School Board of Greater Montreal has recently adopted its calculator policy (encouraging use of calculators at all levels, including 1-3 and L.D. classes).

- Hamilton: Michael Silbert issued a report and recommendations in 1977 on the Handcalculator and Its Impact on the Classroom.

- North York: In 1976, Campbell and Virgin reported on An Evaluation of Elementary School Mathematics Programs Utilizing the Mini-Calculator.

- Edmonton: The Edmonton Public School Board has developed an extensive resource manual for teachers in the elementary shcool: Calculators in the Mathematics Classroom.

Faculties of Education reported that very few courses are available for pre-service teachers, but calculator units are included within existing 'Methods' courses. A number of courses and workshops are offered in In-service programs, and among personnel mentioned where Drost (Nfld.), Gaulin, Mura, and Craig (Que.), Vervoort, Siebert, and Connolly (Ont.), Neufeld, Holmes (alta.), Szetela (B.C.).

Shirly McNicol

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EXTENSION OF A PROJECT IN ENGLAND

During the 1979 CMESG conference, the working group on calculators examined the report

> "A Calculator Experiment in a Primary School"
> by A. Bell, H. Burkhardt, A. McIntosh & G. Moore
> (Shell Centre for Mathematical Education, University of Nottingham, 1978)

of a project which took place in a primary school. An extension of that project has now been initiated in 13 Elementary schools and 2 High Schools, and it is currently entering its second year.

The principles on which the Extension Project is based are:

1. The calculator in the elementary school is an educational aid, rather than a tool children should learn to use.
2. The immediate aim should be to use the calculator to aid the present curriculum: only later will changes to the curriulum be considered.
3. Parents must be kept informed at all stages of the purposes and effects of the work, through lectures, workshop sessions and information sheets.
4. All teachers in the schools are invited, but none are compelled, to be involved.
5. Worksheets and small units should first be produced in response to the teachers' demands for material in specific areas and circulated to all teachers in the schools.
6. When a substantial number of these have been produced, they can be organized to indicate relevant content areas, ages. Gaps can then be identified and filled.
7. Emphasis is placed on observing, assessing and recording the reactions of teachers over time.
8. The end-product should be a useful collection of calculator material and a report.

Up to now, we have reached the following preliminary conclusion:

> Calculators do not change the classroom habits of teachers, although at times they threaten them. If calculators are to be used by the majority of teachers, then they will be used in a variety of ways corresponding to the perceived purposes of mathematics teaching. Major take-over programmes will produce a back-lash if they ignore this.

Alistair J. McIntosh

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CALCULATOR ACCURACY

The purpose of this presentation is to draw attention to a few questions that are generally neglected by a majority of calculator users (including both students and teachers), and yet would be quite valuable in improving both their competence in the use of calculators and their understanding of some mathematical ideas.

The first question is: what are the numbers a calculator displays and works with? The reason why such a question is usually neglected, is probably that it doesn't even occur: the digits and the operation signs of the keyboard look familiar, and so do the numbers appearing on the display. Once the question is raised however, it doesn't take long to establish that the calculator works with a finite subset of the rational numbers. Moreover these "calculator numbers" are not homogeneously distributed: they are denser around zero, while becoming more and more sparse as you move away from zero towards either the largest (positive) or the smallest (negative) number in the system.

After realizing that the calculator universe cannot include every real number, the next question is: how does a real number get translated into a calculator number? This leads to the familiar topics of truncating and rounding off It is a good opportunity to discuss the extra digits carried by many types of calculators (e.g. calculators with an 8 digit display that actually compute on 10 digit numbers).

The third question is: what are the properties of the calculator number system? For any given type of calculator, a list of examples can be devised showing that the calculator number system lacks most of the properties one would expect it to have: it is not closed under addition or multiplication, neither operation is distributive and so on.

This may be quite amusing to someone having unconsciously assumed all along that the calculator arithmetic was the same as the usual one. This exercise may also be of pedagogical interest in that, by asking to look for peculiarities in the calculator arithmetic, it draws attention to properties of the usual number systems that would otherwise go unoticed because too familiar.

Once it has been established that the calculator arithmetic is different from the usual one, the next question is: how much different? In other words: how accurate is the calculator? In answering this question, the notions of error and of relative error are introduced. After pointing out that relative errors are usually quite small, one can give examples of situations in which they are not. It is also interesting to give examples of situations in which even a small error can make a big difference (e.g. trying to decide whether 539 is a divisor of 66 541 707 by executing the division on a 4 function calculator, can lead to the wrong conclusion).

The question of calculator accuracy also gives an opportunity to point out that the accuracy of a result depends on the accuracy of the data as well as on the accuracy of the computation: a result may contain meaningless digits even if the computation hasn't produced any round-off error.

The above questions are designed to make calculator users aware of the practical, and therefore approximate, nature of calculator computations. All the topics mentioned are quite familiar to anybody having taken a course in numerical analysis, however, they are not common knowledge among everybody else, and this includes the majority of mathematics teachers.

Calculators have created a situation in which it has become important that a much larger number of people get acquainted with a few basic ideas of numerical analysis.

Robert Mura

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ABSTRACT FROM "Working Paper on Hand-Held Calculators in Schools"
(M. Suydam, March 1980)

III. Synthesis of National Reports

As one reads the reports in the previous section, one is struck by the similarity of the issues and concerns about hand-held calculators that are being faced in the 16 countries. Some divergence is found, however, in how the issues are being resolved in various countries.

1. Trends, predictions, and prevailing opinions about curricular implications of calculators

"New ideas have often met with public resistance," as Szetela notes (p. 19). Calculators were quickly acquired and are in common use by scientists, engineers, economists, and other professionals; almost every household has one in some countries (e.g., Switzerland, the United States). Yet, although calculators have been in some schools since 1973 (e.g., in the United Kingdom), many of the reports indicate that mathematics teachers may be reluctant to use them. [They generally have been welcomed by teachers of other subject matter, especially the sciences, where calculators are often considered a part of the student's expected equipment (e.g., in Australia), and in non-academic courses (e.g., in Japan)]. Moreover, there is increasing resistance as grade level decreases; that is, primary teachers are most resistant to their use, while secondary teachers have accepted them to a greater extent, especially as a calculational tool, and university teachers allow their use with almost no concern (except in some instances where they ignore calculators, as was noted in Brazil). A quotation from the United Kingdom expresses the picture at the lower elementary level:

> ... the attitude of primary school staff to the calculator ranges from an enthusiastic welcome through passive tolerance to hostility. (Bell et al., 1978 cited on p. 56)

Fielker adds:

> While some are willing to look into the possibilities, the majority are worried that arithmetical skills will be forgotten. (p. 56)

Cheung notes that, in Hong Kong,

> In the primary schools, the use of the calculator is not formally and widely accepted in the classroom because both parents and teachers fear that the calculator might become a crutch to children of this age range and thus impair the learning of the basic computational facts and skills. (p. 25)

Shimada adds that, in Japan,

> Generally speaking, it seems that many teachers are reluctant to introduce calculators in their classes, partially because of their belief that at the elementary school level teachers must concentrate on fundamentals and ... students must master basic skills in computation without special aids, and partially because of monetary problems involved and a fear of new change. (pp. 36-37)

Also from Japan comes the comment that at the secondary level, teachers are "theory-oriented" rather than "practice-oriented" and therefore keep numerical complexity at a minimum; thus, they do not feel the need to use calculators.

In West Germany, it is similarly felt that calculators should not be used before or during the development of computational skills. The calculator is allowed only after grade 7 or 8 in the official regulations of all 11 ministries of education.

Szetela notes that in Canada

> School boards are keenly aware of the objections and concerns of parents toward calculators and hesitate to pursue an uncharted course utilizing calculators without strong public support. (p. 19)

In the United Kingdom, Fielker reports that

> Furthermore, some employers have been vociferous in their demands for arithmetic without calculators. It is sometimes difficult to persuade teachers that calculators are not detrimental to arithmetical health, in spite of continuing evidence in the U.K. and from abroad that the use of calculators improves ability at computation. (p. 56)

And it is noted that in Canada, it is believed that "Calculators should be used to supplement rather than supplant the study of necessary computational skills" (p. 21).

In countries like Thailand, the issues have "not been seriously considered" yet since calculators are less available than in some other countries. New Zealand provides another instance in which there is little argument about the use of calculators:

> Little concern has been expressed by teachers, parents, or the community either for or against the use of calculators ... There has been no strong pressure to introduce calculators into the curriculum, nor to exclude them. (p. 40)

Use in schools tends to be on an informal basis, depending on the initiative of individual teachers within an individual school.

Nevertheless, a statement from Australia is indicative of the status in most countries:

> During 1975 and 1976, both individual teachers and education systems began to realize that they would have to come to terms with the calculator, and that process of adjustment is still underway. (p. 5)

Over and over, similar arguments are raised for using or not using calculators at various schools levels (but especially at early levels). Thus, among the points favoring the use of calculators as a teaching-learning device are:

- attainment of more time for "genuine mathematical content" including new content, and for teaching concepts
- emphasis on problem-solving strategies and mathematical ideas rather than routine calculations
- use of more practical examples and problems with realistic data
- support for heuristic and algorithmic processes
- increase in motivation
- enhancement of discovery learning and exploration
- attainment of speed and accuracy, with relief from tedious calculation
- enhancement of understanding
- lessening of the need for memorization

Most telling of all is the comment that, since their role in society is increasing, students should learn how to use calculators.

The points cited for not using calculators involve:

- fear of dependence on calculators as a crutch, which will "damage the development of children, making them less capable of intellectual achievement"
- tendency to fail to criticize calculator results
- non-availability to all students, since they cost too much for some (thus enhancing the gap between "haves" and "have-nots")
- creation of a false impression that mathematics is computation
- insufficient research on long-term effects
- reduction of achievement in basic computational skills
- decrease in understanding of computational algorithms
- lessening of ability to think
- lessening of ability to memorize
- reduction of motivation to learn computational skills and mathematical principles, or to think through mathematical problems

It is obvious that fear of loss of computational skills with paper and pencil is the predominant concern.

2. Research activities with calculators

The amount of research evidence being accumulated on the effects of the use of calculators varies across countries. In some, there is little or no research activity (Australia, Austria, Hong Kong, Ireland, Japan, New Zealand, Switzerland, and Thailand seem to fit this category). In several others, a limited amount of research is being conducted by college or university faculty and/or graduate students, usually with the focus on the secondary school level or occasionally the upper elementary level (Belgium, Brazil, Canada, Israel, and West Germany appear to fit this category).

In the remaining three countries, the form of research is more extensive, but still varied. In the United Kingdom, the Durham Education Committee, the School Mathematics Project, and the Shell Centre for Mathematical Education have been most heavily involved. Experimentation has been largely exploratory and informal, with the emphasis on ascertaining what could be done with calculators and on the development of comparatively short curricular sequences, with formal data-comparison studies at a minimum. In Sweden, the activity has been organized in a somewhat comparable but more centrally controlled fashion. The first directives from the Board of Education came in 1975; this past year some trials started at various grade levels from 4 through 12. ARK (Analysis of the Consequences of Calculators) is coordinating a wide program of research and development, studying the effect of calculators as an aid for calculation, as an aid in changing the methods in the present curriculum, and as an aid in changing the content of the present curriculum.

In the United States, over 100 studies have been conducted, most independent of the others and most using an experimental design in which the achievement of calculator and non-calculator groups on various curricular topics or modes of instruction have been compared. Almost all such studies indicate either higher achievement or comparable achievement when calculators are used than when they are not used. A handful of studies has looked at learning-oriented quesstions, in an attempt to ascertain how learning of mathematical ideas (rather than merely achievement) can be improved with the use of calculators. In addition, less-formal curriculum development studies have been underway, to develop sequences for instruction at most grade levels. The majority of such work has concerned integration of the calculator into the existing curriculum; far less attention has (thus far) been given to revising the curriculum to integrate calculators. This research and development work is proceeding at several levels, with university mathematics educators, supervisors of mathematics, and teachers collaborating in some attempts and working largely independently in others. While federal funding has supported some studies (including a number of grants directly to school systems), other work has proceeded solely because an individual (including many doctoral candidates) felt the need to pursue the topic and collect evidence.

Across countries, the overwhelming majority of the data -- from both formal experiments and informal experimentation -- has supported the conclusion that the use of calculators does not harm achievement scores (in particular, computational scores). From the few studies focusing on the point, there is indication, in fact, that use of calculators can promote computational skill achievement, as well as the learning of other mathematical ideas.

There is continuing need for informal research by teachers at the classroom level to establish and evaluate ways of using calculators to investigate both existing and new topics in the curriculum. Similarly, mathematics educators need to continue efforts to ascertain how the calculator can serve best as a tool to promote learning.

3. Instructional practices with calculators

The need for curriculum development is evident in the comments from several countries; for example, a comment from Austria indicates

> A main obstacle for intensifying the use of calculators is the fact that neither the curriculum of mathematics nor the schoolbooks are related in any way to the needs and possibilities of the calculator. (pp. 11-12)

The Australian reports adds:

> For the mathematics classes, there has been a more systematic consideration of how calculators might be used, not merely to carry out calculations ... but also to attack new types of problems, or old types of problems in new ways. (p. 6)

In the report from Ireland, it is noted that

> Teaching methods and course content may need to be revised to take advantage of calculator techniques. (p. 29)

And the Canadian report points out that

> ... teachers would prefer to wait [to use calculators] until materials ... are written. (p. 21)

"Restrained enthusiasm" and a "lack of bold projects for implementation" are also noted (p. 23). There is need to incorporate additional work on such topics as estimation and approximation, significance of answers, mental arithmetic, rounding, and flow charting; the limitations of calculators also need to be taught.

In most countries, as the report from Belgium notes,

> There is still some quarreling going on between two groups, the one stimulating the use of the calculator only as a computational tool and the other willing to take more advantage of calculators as an instructional aid. (p. 13)

It seems fairly evident that, given the concern by parents and teachers about students learning paper-and-pencil computational algorithms, these will not rapidly disappear from the curriculum in most countries. Fielker notes that

> Even among the more enthusiastic schools, the calculator has been assimilated into existing curricula, and no one so far has altered the curriculum to take account of the calculator. However, there is enough development taking place to indicate the way things eventually should go. (p. 56)

He indicates that paper-and-pencil algorithms will persist as only part of an "armoury of techniques". Pupils will design their own algorithms, and the focus will be on the development of algorithms for calculators and computers. Thus, he writes:

> ... one questions whether ... written arithmetic is any longer necessary. ... What is required is an ability to check that answers are reasonable; hence a facility with "single digit" arithmetic, and a sense of the size of numbers to be expected in real situations [is needed]. (p. 57)

In Israel, similar concern is expressed for algorithmic thinking, but the focus is

> not to try to revolutionize the mathematical curriculum, but rather to modify it by putting emphasis on algorithms for solving meaningful problems and carrying them out to completion. (p. 31)

From Japan comes this comment:

> If we admit an assumption that calculators are always available in any classroom, then emphasis might be shifted from mastering a certain set of prescribed algorithms to developing various algorithms based on fundamental properties of number, and grading difficulty by number of digits in computation might become not so meaningful. (p. 38)

However, few textbooks have incorporated use of calculators.

Sweden is relatively advanced in curriculum development compared with most other countries. New curricular sequences integrating calculators are being tested. In West Germany, Brazil, Argentina, Israel, the United States, and the United Kingdom, there are some smaller-scale efforts to develop calculator-integrated curricula. While the Swiss report indicates that "changes in curricula are for the time being not necessary because of the use of calculators" (p. 47), curricular guides may include use of calculators as aids; such statements have also been included in many other countries. Thus, the official syllabus for New Zealand recommends the use of calculators, but only at Form 7 (for 17- and 18-year-old students). In Hong Kong, the new mathematics syllabus for secondary schools attempts to incorporate the use of calculators, including decreases in emphasis on some topics and the addition of several new topics. In most countries reporting recommendations, the use of calculators is suggested after grade 7 -- that is, after initial teaching of computational skills is completed and they have presumably been learned.

A matter of concern to many persons -- though at a lower level than the concern over use of calculators in elementary schools, associated with fears over loss of computational skills -- is the concern over their use on tests. The report from Ireland provides a particular illustration of this. Calculators were used on examinations in 1974 and 1975. Then:

> Public awareness of a possible social discrimination in the use of calculators in examinations was fanned into life by an article in one of the leading daily newspapers; the Minister of Education became worried, and calculators were banned from the public examinations in 1976. They have been banned since then . . . (pp. 28-29)

In Australia, school inspectors began in 1977 to recommend that students be permitted to use calculators on public examinations. By 1980, virtually all examining boards will permit the use of calculators on grade 12 examinations. From 1980 on, the calculator will also be allowed in the Hong Kong Certificate of Education examinations.

By 1978, four of eight boards administering the General Certificate of Education Examination in England were permitting use of calculators on Ordinary Level, and two others permitted them at A Level; the Scottish Board was permitting them in all examinations. However, they are not allowed on the English Certificate of Secondary Education examinations (and therefore most schools have not used calculators with average and below-average pupils).

In New Zealand, on the other hand, some locally based regional examinations for low achievers in mathematics do incorporate the use of calculators, although they are not permitted in national school-level examinations. Calculators (including programmable types) are allowed in secondary-level examinations in Sweden, but not in grade 9. The policy in West Germany is that calculators are allowed on tests in grades 8-13 whenever calculation is not a goal of the test (calculators are not used below grades 7 or 8 in the instructional program).

In the United States, calculators are not allowed on standardized tests (because of their construction and norming), and most teachers do not allow their use on any mathematics tests. The College Entrance Examination Board is considering the need to modify existing tests or develop new tests on which calculators will be used, however. Their actions could have an impact both on other test developers and on the use of calculators in instruction.

4. Student outcomes, attitudes, and concerns about calculators

Students are generally positive about using calculators. Some teachers (in Belgium), however, reported that students quickly lose interest "when they realize that much thinking is involved in working with them" (p. 14). Whether for this reason or another, the initial high level of motivation that is usually attained when calculators are first introduced is rarely a lasting phenomenon. (After all, the calculator is a tool; consider the chalkboard: does it keep children excited day after day?)

It may well be, however, that low achievers who have continuously failed in mathematics may find that the calculator provides a means to help them succeed. And as the curriculum changes, some anxieties about mathematics may be relieved, causing a long-lasting motivational effect.

5. In-service activities on calculator use

Little cohesive planning of in-service activities for teachers, to help them place the calculator into perspective and develop strategies for using it effectively, was reported. Professional associations of teachers, local or national educational authorities, teachers' centres, and universities have shouldered the burden of sponsoring conferences, meetings, seminars, workshops, short courses, and/or discussion groups. There has been little systematic planning for these, however: they seem to arise upon demand, with little attention devoted to creating a demand so that teachers will be prepared to cope with calculators (and other technological phenomena which could impact upon schools).

Articles have been published in journals in most (but not all) countries reporting, although the number of such publications, and of books, shows a wide variance from country to country. In the United States and West Germany, information centers provide an additional means of disseminating materials to teachers (and to others interested in the use of calculators).

As was noted in the Canadian report, the in-service activities are "not as widespread as needed" (p. 22) and "it can hardly be said that [calculator materials] are sought by a tidal wave of teachers" (p. 22).

6. General background on amount and type of use of calculators

Three generalizations about calculator use seem appropriate:

(a) The higher the age level, the more likely students are to own or have access to calculators.

(b) The higher the grade level, the more likely that calculators are used in schools.

(c) Four-function calculators are most used in elementary schools, scientific calculators in secondary schools, and programmable calculators by the college years. Calculators with algebraic logic appear to be most widely used (especially in pre-college years), probably because they are more widely available at lower prices.

Calculators are either bought by individual pupils (sometimes through school-organized plans) or by schools. In Japan, the Ministry of Education subsidizes one-half the cost of equipping schools with calculators, according to a schedule, and limited federal funds are available in some other countries (e.g., the United States) for purchasing calculators.

Concluding Comment

The terms "fluid situation" and "cautious approach" appear in the reports, and would seem to characterize the situation in many countries. In at least one country, New Zealand, "Other issues in mathematics education

... appear to be of greater priority" (p. 41).

The Australian report notes that the need to preserve a reasonable arithmetical facility will continue to be argued. But, as Fielker notes:

> Unless we effect the necessary changes in educational attitudes, it could be that the classroom will be the only place where arithmetic is done by hand! (p. 58)

CMESG/GCEDM 1980
Review Group II

MATHEMATICS EDUCATION RESEARCH
WHAT IS IT ABOUT IN CANADA?

Thomas E. Kieren*
University of Alberta

Mathematics education research like all research represents a quest for knowledge. What is this knowledge about in Canadian mathematics education research?

In exploring this question it is useful to cite four recent statements about knowledge. Bateson (1979) maintains that useful knowledge must be about something. In seeking knowledge, or in any mental functioning for that matter, Hofstadter (1979) suggests that one can function in a mechanical or an intelligent mode, but human beings usually try to bring meaning to mechanical functioning. Such meaning is developed from isomorphisms with reality and relys on symbols as mental triggers. Finally persons should build up constructs about realities which are extensive and connected (Margenau, 1961).

These comments have parallels in mathematics education.

1. Useful mathematics education knowledge must be about something.
2. Mathematics education researchers can function mechanically or intelligently, but tend to bring meaning to their work.
3. Meaning in mathematics education comes through isomorphisms to appropriate reality.
4. Mathematics education ideas should be extensive and connected.

How can mathematics education research be congruent with these statements. At the 1979 CMESG meeting Easley discussed the relationship between the researcher's space of interest and the mathematics education reality for the teacher or student. When the projection of the research space has only limited contact with teacher-student reality, such research has two problems. Philosophically, it is not about that which it should be.

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* John Travers of the University of Alberta helped in preparing the survey here.

Practically, its results can be safely ignored by teachers. Easley admonishes researchers to so develop their research, that it projects onto a significant area of practice.

Bauersfeld (1976) discusses mathematics in education in terms of matter meant, matter taught and matter learnt. If one considers each of these as a vertex of the triangle of mathematics research, such research would focus on mathematics curriculum development and analysis, mathematics instruction and mathematics knowing, learning and developing respectively. To follow Easley's direction the center of this "triangle" would be student mathematical experiences and mathematics classroom experiences.

In terms of this research triangle one can make a number of observations with respect to mathematics education research. First, as would be true anywhere, because of its proximity to classroom experience, the research and development efforts of teachers is both most prevalent and probably most important. This research, generally informal in nature, is informed by the classroom actions of students. It can and under Easley's model likely will be informed by more formal mathematics education research. Some of this research occurs in natural classroom settings, focusing on learners in action or teachers in action. This research often uses participant observer or stimulated recall methodologies. While there is a quantity of such research in Canada using mathematics classrooms, much is done by persons interested in administrative or personal aspects; hence such research while informing mathematics education does not generate mathematics education knowledge.

One form of curriculum research prevalent in Canada is evaluation. This work generally occurs under the aegis of the various provincial governments, but has been extended in a variety of ways by researchers particularly in British Columbia as well as in Alberta and Ontario.

Another form of research which has a long history in Canada is the analysis of mathematics with the purpose of providing a sound basis for curriculum development. Such research has been done recently with respect to several aspects of geometry, rational numbers, and the real plane. In conjunction with evaluation studies, calculus in school mathematics also has been considered.

There have been a number of recent "learning-knowing" studies in Canada. One area of emphasis has been problem solving. Studies of processes used in solving non-standard problems have been considered both for young children (3 - 8) and high school students. The heuristics used by high school students have also been considered.

There have been a number of studies based on the observation of children and young adults working on tasks which might be considered subject oriented. The subject areas of such tasks are ratios, rational numbers and algebra.

One "learning-knowing" study was related more directly to instruction. In this research, students in an active project oriented environment in ratio and rational numbers were studied. A number of comparative performance differences were noted favouring a process oriented approach.

Another kind of learning-knowing study, which was a focus of a working group at the 1980 CMESG meeting, pertained to diagnosis. Here the object was to identify student performance difficulties and suggest alternative teaching interventions. Another teacher related study along this line is on-going and its objective is to observe the effects of training teachers in the use of clinical methods.

Finally there is a body of recent Canadian research on teachers and teaching. Much of this research focuses on teacher characteristics and not surprisingly much of the work has been done with pre-service elementary education majors. There has been work on the mathematics anxiety of teachers and also how this can be alleviated. There is research on pre-service teacher performance in mathematics related to various methods course settings as well as on pre-service teacher attitudes.

In addition to the rational number study mentioned above some research has focused on methods of instruction. Some of this research, which also has a curriculum aspect, has considered the impact of calculators and to a limited extent computers, on instruction at various levels.

Attached to this report is a sampling of recent or on-going research in Canada. Some information came from a survey of recent Canadian publications. Other material was produced as a result of a request for information about on-going research. It is appended here in the form found or received. This survey is in no way exhaustive, but does give an indication of recent Canadian mathematics education research activity.

The Canadian Mathematics Education Study Group has as a goal to keep its members informed about on-going activity. In addition, through its meetings and stimulated interaction of members, it can prod research into being better projected onto the world of mathematics student and teacher at many levels. Thus such research has a better chance of generating mathematics education knowledge which is useful, meaningful, extensive and connected.

CURRICULUM RESEARCH

Dieter Lunkenbein; Universite de Sherbrooke. "Le concept de groupement de Jean Piaget comme outil de la rationalisation des interventions didactiques"

Content analysis - instructional development - mathematics learning and development.

Study of the relevance of the notion of grouping as an organising structure of conceptual contexts, as a model for the description of mental images of the learner and as an element to determine strategies for teaching interventions.

Fernand Lemay; Universite Laval, Québec. Une étude structurale du comptage; mathematics learning and development.

Approche de l'arithmetique dans une perspective ordinale. Au moins 6 niveaux d'apprentissage sont distingués. Le comptage emerge comme "object d'exploration". Vision rénovée des algorithmes de calcul (Textes disponibles).

Fernand Lemay; Universite Laval, Québec. Genèse de la géométrie; mathematics learning and development.

Poincaré a parlé de la géométrie projective comme étant la "géométrie de la lumière"; cette vision est poursuivie avec acharnement. Plusieurs parties réalisées: "Le projet de la g.", "Germes de l'activité géométrique", "Etincelles et groupe projectif", . . . "Synthèse géométrique des nombres", . . .

Fernand Lemay; Universite Laval, Quebec. Modèle réel du plan projectif complexe; mathematics learning and development.

Vers un nouveau modèle, peut-être plus "intuitif" que celui de von Staudt.

Fernand Lemay; Universite Laval, Québec; *Géométrie projectives finies associées aux polyèdres platoniques*; mathematics learning and development.

Aspects exemplaires d'enseignement de la géométrie par les géométries finies.

Doug Crawford; Queen's University.
Basic Skills and Understanding in School Mathematics; Evaluation Study, Mathematics Skills and Learning.

Two related studies are underway. The immediate objective is to make a survey of national and state-wide assessments of competency in basic skills in school mathematics. These include the first and second National Assessments in the U.S.A., the assessment in England under the supervision of the Department of Education and Science, and similar assessment movements in Canada, particularly in Alberta and British Columbia. Analysis of such surveys will then be made to identify areas of strength and weakness, so that learning and teaching of problem areas can be improved.

A second objective is to compile an inventory of those skills deemed basic by the mathematics education and teaching community. This will be done by analysing the make-up of the various surveys, and sampling the views of specialists and organizations.

A third and more long-term objective is to define more adequately what is meant by a "mathematical skill" and to relate the acquisition, of skills in selected basic areas of mathematics, e.g., computation, estimation, and approximation the level at which the mathematics is "understood", thus linking the study to the on-going research of Dr. R.R. Skemp on "intelligent learning" and its relationship to mathematics learning.

Bramwell, J.R., Vigna, R.. Ministry of Education, Ontario. Evaluation Instruments Locally Developed in Ontario; annotated catalogue.

Describes evaluation materials developed by testing personnel, curriculum specialists, researchers and teachers from across Ontario. Seven major types listed: Achievement tests, Achievement tests - Diagnostic, Attitude scales, Behaviour rating scales, Personality tests, Observation checklists, Questionnaires and surveys. For Mathematics there are 64 Achievement tests listed, 11 Diagnostic tests, 1 Observation checklist, 1 survey.

Assessment Instrument Pool (OAIP), Ministry of Education, Ontario; evaluation study.

For distribution in fall of 1980. Intent is to provide assessment instruments ("test items") directly related to the objectives of Ministry guidelines. Loose-leaf binders will be sent to each school. These will contain: assessment instruments, codes for indexing them to objectives, examples of student writing, province-wide standards in certain areas, material to act as standards on such affective aspects as listening, speaking and writing. There are expected to be more than 4,000 test items for mathematics.

Al Olson, University of Alberta; Geometries.

Various extensive curricular analyses involving non-metric measure (decomposibility) and the geometry of shape. Curriculum materials for junior high students have been developed and tested.

Some Recent Publications

1. Murray McPherson

   What do Teachers and Pupils Think About the Mathematics Programs in Manitoba.

   Man Math T 6:1:10-11 O'77

2. Achievement in Basic Skills: A Longitudinal Evaluation of Pupil Achievement in LA and Mathematics (Review) - H.G. Hedges, CJED 4:1(1979).

   g 5 & 8 over 40 years - 1976-1938-1954

   a. average grade level in 1975 was higher than that of students in the 1930's and 1950's.
   b. g 8 students in 1975 performed consistently less well in arithmetic computation and reasoning.
   c. g 5 to 7 today outperform in fundamental operations .

3. Achievement Results from the B.C. Mathematics Assessment - David F. Robitaille, James M. Sherrill, CJED V4, No. 1, 1979, p 39.

   March 1977 - 100,000 students - grades 4, 8, 12

   Achievement relatively good - computational skills, knowledge of terminology, and at grade 4 and 8 problem-solving.

   Areas needing improvement - comprehension of fraction concepts, comprehension of geometry and measurement concepts, and problem-solving at the senior-secondary level.

4. James M. Sherrill and David F. Robitaille
   Comments on the Mathematics Assessment Results: Problem-Solving
   Vector 19:2:17 - 31 D'77

5. James M. Sherrill
   Provincial Learning Assessment Program - Geometry.
   Vector 19:3: 13-38 Mr'78

6. Victor Steblin and Ian D. Beattie
   Math Education in B.C. Local Curriculum Development: Questionnaire Results
   Vector 19:1:8-14

7. David F. Robitaille and James M. Sherrill
Provincial Learning Assessment Results: Highlights
Vector 19:1:19-24 O'77

8. Sue Harberger
Report of the BCAMT Senior Secondary Questionnaire (concerning math education in B.C.)
Vector 18:4:5-8 Jn'77

9. John R. Bramwell and Roxy Vigna
Evaluation Instruments Developed Locally in Ontario
Ministry of Education (Ont.) 1979

10. E.W. Surgenor and V.H. Fick
Relating Criterion Referenced Tests to Specific Learning Outcomes (Interim report)
Educational Research Institute of B.C. 1978 (Rep #78:23)

11. Alberta assessment of school mathematics. Condensed report. A.T. Olson, D. Sawada and S.E. Sigurdson. Edmonton: Alberta. Minister's Advisory Committee on Student Achievement 1979. 94p bibl. tabs.

12. David F. Robitaille
Provincial learning assessment program - instructional practices
Vector 19:2:13-23 Jn'78

13. John V. Trivett
Forward to the basics
Vector 19:3:25-38 Mr'78

TEACHER AND INSTRUCTION RESEARCH

1. D.J. Bale, University of Regina. An Exploratory Study of Computer Aided Problem-Solving. (Instructional Development/Math Learning)

   The purposes of this study are as follows: 1. To write a program that can be used by a person trying to solve a math problem in the interactive mode; one that helps the student by reacting to his attempts with Polya type hints. 2. To study patterns of problem-solving by interpreting the computer printouts of these problem-solvers.

2. Jim Vance, University of Victoria.

   Handheld Calculators and Percent - Curr. In., evaluation study.

   A calculator-assisted unit on ratio and percent was taught to a group of Grade 7 students. Calculators do not reduce the amount of instruction needed on basic concept, estimation, and problem-solving skills. Calculators do permit students to solve and investigate a greater range of verbal and non-routine or open-ended problems involving extensive computation. Student attitudes were positive.

3. M.P. Carroll, J. Gershman, E.J. Sakamoto; O.I.S.E., CARE CAI Network, Intermediate Mathematics; Instructional development.

   A CAI network involving 52 terminals in 21 schools. Courses are all at the Intermediate Level (Grades 7 to 10). Topics are: Arithmetic: Whole Numbers, Fractions and Percent, Integers and Rationals, Applications; Algebra: Algebra, Equations; Measurement; Probability. French and English versions are available. Branching strategies, interactive lessons and games are features.

4. Harrison, D.B., S. Brindley, M. Bye, University of Calgary.

   Instructional Development: Study included the teacher-aided development of an activity-oriented instructional package in Grades 7 and 8 (ratio and rational numbers). It assessed student achievement as well as the effect of the level of congruence between student thought levels and instruction used.

5. Werner Liedtke, University of Victoria.

Instructional Development: Observational Study - Case Studies - Slow Learners in the Early Grades

Curriculum Development: Observational Studies - Games/Game playing behavior: preschool children - Problem-Solving Settings for Preschool/ Kindergarten..

6. D. Bale, University of Regina.

Profiles of Elementary Education Students classified as Math Anxious. (Math Learning (variables)).

Students given MARS and classified as Math anxious are being examined for other characteristics and profiles are being drawn up in search of patterns. Mathematics attitude, aptitude, mathematics background and some personality traits are being measured.

7. I. Burbank, University of Victoria.

Evaluation Study. (Data collected. Now being processed and analyzed)

Studying Following Relationships: (Elementary Teach.)

1. Math. Content Scores and General Math. Assessment
2. " " " and Practice Teaching
3. " " " and Math. Attitude Scale
4. " " " and Math. Method Scores
5. " " " and Reading Assessment Scores
6. " " " and Grammar Assessment Scores
7. Math. Assessment Scores and Practice Teaching Scores
8. " " " and Math. Attitude Scale
9. " " " and Grammar Assessment Scores
10. " " " and Reading Assessment Scores
11. " " " and Math. Methods Scores
12. Practice Teaching Scores and Reading Assessment
13. " " " and Grammar Assessment
14. " " " and Math. Methods Scores
15. Math. Attitude Before Math. Methods and
    " " After Math. Methods

Comparison of Secondary Teachers and Elementary Teachers on

1. Math. Assessment
2. Math. Attitude
3. Grammar Assessment
4. Reading Assessment

Comparison and Relationship of:

1. Regular Students who Took 160 and Their Practice Teaching Score
2. Transfer and Diploma Students who Took Math. and Their Practice Teaching Score

Some Recent Publications

1. W. Higginson

   Book Reviews & Research . . Research and the Classroom Teacher

   DMT Math. G. 16:1:50-51   5'77

2. T. Holens

   Report of the Committee on the Use of Calculators in the Schools

   Man Math T 6:2:16-17 Ja'78

3. James M. Sherrill

   Survey on the use of hand-held calculators in mathematics classes in secondary schools of British Columbia

   Vector 18:4:9-10 Jn'77

AJER

4. J.H. Vance

   Attitudes Toward Mathematics and Mathematics Instruction of Prospective Elementary Teachers

   AJER V24, No. 3 p. 164

5. Renee Caron

   Les attitudes et l'apprentissage de la mathematique

   Ont. Math. G 17:3:28-32 Mr'79

6. Jean-Paul Collette

   Measure des attitudes des etudiants de college 1 a l'egard des mathematiques

   Ministere de l'Education, 1035 de Lachevrotiere, Quebec G1R 5A5

7. Olson A.T. and

LEARNING-KNOWING RESEARCH

J. Hillel, D. Wheeler; Concordia University; Problem Solving Processes; content-analysis-instructional development.

Analysis of problem-solving processes of approximately 100 high school students solving non-standard mathematical problems while 'thinking out loud'. Question: What does one learn from individual interviews which can be transferable to classroom problem-solving activity.

Nicholas Herscovics, Universite Concordia; Jacques C. Bergeron, Universite de Montreal;

The Training of Teachers in the Use of Clinical Methods

J.D. Burnett, W.C. Higginson, G.L. Hills, H. Osser and M.G. Schiralli; Queen's University;

Mathematical Knowledge: A Study of Cognition and Metacognition

Mathematics knowledge, school mathematics, clinical interview.

A series of studfies on how students attempt to come to terms with school mathematical knowledge. All of the studies used a form of clinical interviewing. The children studied were from grades 7 and 10.

Reports:

Burnett, D. Educational Psychology and Mathematical Knowledge: An Analysis of Two Student Protocols, 1978. 207 p.

Hills, G.L.C. Through a Glass Darkly: A Naturalistic Study of Students Understanding of Mathematical Word Problems, 1978. 37 p.

Osser, H. Making Sense of It? Trisha in the World of Bromdas, 1978. 15 p.

Schiralli, M.G. Language and the Child's Understanding of School Subjects, 1978. 27 p.

Burnett, J.D., Queen's University. A Structuralist Investigation of Students' Mathematical Knowledge; mathematics knowledge, school mathematics, clinical interview.

Twenty-four students from two schools in N.S.W. Australia (Grades 7 and 11) were studied using two variations of a clinical interview. Complete protocols are available and are presently being analyzed for evidence of student conceptions of topics presented as part of the schools' mathematic curriculum.

Tom Kieren, University of Alberta. Rational Number Thinking in Children and Adults; content analysis - mathematics learning and development.

A fractional number thinking test based on four rational number interpretations was devised and 650 Grade 7 and 8 students tested. A similar thinking pattern was observed in all interpretations, with very strong differences in unit and non-unit fraction task performance observed.

D. Sawada, University of Alberta. Cross-modal matching and number perception. Studies the reaction of early school children to mathematical (numerical) information presented visually, aurally or haptically.

L.D. Nelson, Problem-Solving in Young Children. A summarizing of many years of research on the acts of young children in a variety of mathematical settings. Available through the office of the Dean of Education, University of Alberta.

Some Recent Publications

L. Pereira-Mendoza, Heuristic Strategies Utilized by High School Students. AJER V25, No. 4, p. 213

Most common heuristics - systematic cases or patterns, symmetry, and analysis. Utilization of heuristics tended to be problem dependent.

T.E. Kieren and B. Southwell, The development in children and adolescents of the construct of rational numbers as operators. bibl. tabs. AJER 25:4:234-247 DEC 1979.

CMESG/GCEDM 1980
Review Group III

VERS UNE INTEGRATION DE LA RECHERCHE A LA FORMATION ET AU PERFECTIONNEMENT DES ENSEIGNANTS*

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SOMMAIRE

L'initiation à la recherche peut valoriser l'enseignant et lui permettre d'innover dans sa classe. Les éléments de la recherche à intégrer dans la formation et le perfectionnement des maîtres doivent être choisis selon les exigences didactiques de chaque discipline. Ceci est illustré dans le cadre de la didactique de la mathématique dont la nature abstraite et formelle pose de sérieux problèmes pédagogiques. Des méthodes de recherche permettant d'observer et d'analyser la pensée de l'éléve sont illustrées par des exemples. Il est suggéré, que pour qu'une intégration rationnelle de la recherche à la préparation des enseignants puisse un jour être faite, des efforts analogues soient poursuivis dans d'autres disciplines.**

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*Article paru dans le numero du printemps 1980 de la Revue des Sciences de l'Education (Vol.6, No.2.)

** L'élaboration de notre version de l'expérimentation didactique soviétique a été subventionnée par l'Université de Montréal (CAFIR, 1978).

1 INTRODUCTION

> *En formation initiale, l'Université doit introduire dans ses programmes une initiation à la recherche... et la recherche doit être l'une des voies organisées du perfectionnement.*
>
> (Commission d'étude sur la formation et le perfectionnement des enseignants. Rapport - mai 1979, p. 60).

Dans son rapport au Ministère de l'Education du Québec, la Commission d'Etude sur les Universités recommande l'intégration de la recherche à la formation et au perfectionnement de l'enseignant afin de lui faire *acquérir des méthodes et des concepts qui lui permettent de rester capable d'innover, de percevoir le changement et de restructurer ses perspectives*. De plus, elle suggère que les types de recherche à développer soient ceux qui *permettent d'améliorer la qualité de l'éducation, la valeur éducative des écoles et la croissance optimale des élèves*. Elle souligne enfin qu'il faut *valoriser et sortir de l'isolement les recherches spontanées des enseignants*.Ces propos justifient le besoin d'une telle intégration tout en évitant d'entrer dans la controverse de ce qu'est la recherche.

On ne peut cependant ignorer ce qu'en pensent les enseignants. Certains croient que la recherche en Sciences de l'Education demeure le domaine du professeur d'Université ou du chercheur professionnel oeuvrant dans un centre de recherche; qu'on y accède qu'une fois inscrit dans un programme de deuxième ou de troisième cycle; que les méthodes de recherche applicables sont celles des sciences naturelles et qu'elles se doivent d'être quantitatives; que les problèmes abordés sont souvent si théoriques et si loin de la réalité que leur étude n'apporte aucune aide pratique. D'autres estiment que certains travaux présentés au premier cycle, dans le cadre des cours réguliers, et dits *projets de recherche*, constituent réellement de la recherche quand souvent ils ne se limitent qu'à un relevé de la littérature.

La Commission interprète la recherche dans un sens plus large que les enseignants et elle semble convaincue que l'initiation à certaines méthodes leur permettra d'innover dans l'exercice de leur profession. Sans chercher à définir la recherche, nous pensons qu'elle se caractérise par une application consciente et minutieuse de certaines méthodologies éprouvées. Il ne s'agit pas d'introduire au niveau du baccalauréat toute la gamme des méthodes de recherche employées par les chercheurs professionnels. Il faut plutôt en identifier les éléments pouvant être assimilés par les futurs-maîtres et utilisés par les enseignants eux-mêmes dans leur classe.

Les éléments qui pourraient être inclus dans la formation de l'enseignant doivent être jugés en fonction des exigences didactiques de la discipline considérée. En effet, chacune d'elles possède ses propres problèmes et ses propres didacticiens (on n'enseigne pas la mathématique comme on enseigne la musique). C'est donc à eux que doit revenir la tâche de déterminer pour leur domaine particulier quels sont les aspects de la recherche qu'ils jugent accessibles et utiles au maître. Evidemment, de la somme de tous ces travaux se dégageront des éléments communs, ce qui permettra une intégration à la fois mieux coordonnée et plus rationnelle.

C'est dans cet esprit que nous décrivons les problèmes particuliers à l'enseignement de la mathématique et que nous déterminons les aspects de la recherche pouvant aider à les résoudre.

2 PROBLÈMES PARTICULIERS À L'ENSEIGNEMENT DE LA MATHÉMATIQUE.

Les recommandations de la Commission s'avèrent des plus intéressantes pour les programmes de formation et de perfectionnement des maîtres en mathématique, vus les problèmes particuliers que pose l'enseignement de cette discipline. En effet, la mathématique étant une science formelle , par opposition aux sciences expérimentales telles que la physique, la chimie ou la biologie, son contenu se distingue difficilement de sa forme de représentation. La communication de concepts mathématiques s'avérant pénible sans l'emploi d'une représentation symbolique, il en est résulté une tendance à se centrer sur le jargon, sur la notation et sur la manipulation de symboles.

Jusque dans les années cinquante, c'est justement ce à quoi se limitait l'enseignement de la mathématique (la mémorisation et la manipulation d'un tas de formules éparpillées). L'essentiel du renouveau des vingt dernières années a consisté en une tentative d'unification des diverses notions enseignées par l'introduction de concepts unificateurs (ensembles, fonction, ...) et de structures algébriques (groupes, anneaux, corps, espaces vectoriels,...). Bien que permettant un développement précis et logique de la matière, un tel programme exige un formalisme et un vocabulaire excessifs (Kline, 1973) au détriment de l'aspect psychologique de l'apprentissage (Skemp, 1971). Il a été trop facile de croire que l'élève assimile un concept simplement en mémorisant son nom et qu'il donne nécessairement un sens aux symboles en apprenant à les manipuler.

De récentes études psycho-pédagogiques indiquent que c'est aux niveaux symbolique et formel que se situent les problèmes d'apprentissage. Carpenter et Moser (1979) ont montré qu'après deux ans d'arithmétique des enfants de deuxième année avaient plus de difficultés que des enfants du préscolaire, face à certains problèmes raisonnés. Ginsburg (1977) lui, a rapporté que des enfants ayant des difficultés en arithmétique écrite se débrouillaient très bien si on les laissait compter sur leurs doigts. Même au niveau secondaire, des étudiants maîtrisant la notion de pente d'une droite au niveau graphique, s'y perdaient avec la formule (Herscovics, 1980). Ainsi, la majorité des élèves semblent pouvoir acquérir une compréhension intuitive d'un concept mathématique tant que sa représentation demeure non formelle. Ce n'est que lors de sa formalisation que des différences dans leurs habiletés pourraient être décelées.

L'enseignant qui ne se rend pas compte de ces difficultés tend à enseigner d'une façon formelle et perd ainsi dès le départ une grande partie de la classe. Par contre, en utilisant des représentations intuitives et une approche relationnelle, il en atteindra certes un plus grand nombre. Pour beaucoup d'élèves, ce n'est que lorsqu'il y a eu un tel *accrochage* intellectuel qu'ils peuvent formaliser les notions mathématiques et se construire ainsi une signification pour les symboles. Une telle démarche implique que l'enseignant doit distinguer entre le contenu et la forme mathématique d'un concept. De plus, il doit pouvoir analyser les observations recueillies auprès des étudiants pour juger si les représentations qu'il a élaborées sont à leur portée. Dans le cas contraire, il doit en inventer des nouvelles, en un mot innover.

3 TENTATIVES D'INTÉGRATION.

En didactique de la mathématique, plusieurs tentatives ont été faites dans différents pays pour rapprocher la recherche de l'enseignement.

Aux Etats-Unis par exemple, le NCTM (National Council of Teachers of Mathematics), la principale association regroupant les enseignants de la mathématique des niveaux primaire et secondaire, publie une revue entièrement dédiée à la recherche (Journal for Research in Mathematics Education). A l'origine, elle visait les maîtres, mais ses articles, exclusivement psychométriques, semblent n'avoir intéressé que les chercheurs. De plus le NCTM organise, lors de ses congrès annuels, des activités touchant la recherche, mais peu d'enseignants y participent.

En France, on avait doté une vingtaine d'Universités d'un IREM (Institut de Recherche sur l'Enseignement des Mathématiques) dont la fonction principale était de recycler les maîtres à l'enseignement des mathématiques modernes. Seuls les IREMs de Bordeaux, d'Orléans et de Strasbourg sont encore impliqués dans la recherche et n'y participent, à titre d'auxiliaires, que quelques enseignants.

En Union Soviétique la recherche en didactique de la mathématique se rapproche de l'enseignement. La plupart des études traîtent d'un aspect ou d'un autre du programme scolaire. Quoique les enseignants choisissent eux-mêmes, selon l'habileté des élèves, ceux qui seront les sujets d'expérimentations didactiques ,ils n'en demeurent pas moins des auxilliaires (Kantowski, 1979).

Ces essais de rapprochement visaient à faire participer l'enseignant à la recherche universitaire (Gaulin, 1978), mais en laissant entier le problème de la transformation de celui-ci en chercheur (au sens large) autonome. Seuls l'Angleterre et le Québec ont été témoins de quelques modestes tentatives un peu mieux orientées dans cette voie. Depuis bientôt dix ans la principale association de professeurs de mathématiques d'Angleterre (Association of Teachers of Mathematics) encourage ses membres à s'impliquer dans la recherche. S'adressant à eux, Wheeler (1970) caractérisait en ces termes l'enseignement *scientifique*:

> *la technique principale de l'homme de science est d'agir sur la situation dans laquelle survient un phénomène afin de la changer et d'observer les changements ainsi provoqués. C'est cette technique qui permet l'étude scientifique du processus d'apprentissage de l'enfant. Elle ne requiert nullement que les phénomènes étudiés puissent être contrôlés ou isolés des situations où elles surviennent. Le rôle du scientifique n'est pas de se retirer afin d'observer, mais d'agir et de continuer à observer.*

Bishop (1971, 1972, 1975) a proposé la notion de *l'enseignant-chercheur*. Pour lui, la recherche c'est *toute tentative systématique de cueillette d'évidences sans nécessairement utiliser des statistiques ou des groupes-témoins*. Parmi les sujets de recherche suggérés aux enseignants, on trouve l'étude de cas, l'enregistrement de leçons sur bandes magnétiques pour fin d'analyse, et l'expérimentation dans sa propre classe de divers types de comportement personnel. Quoique ne touchant ni à la formation, ni au perfectionnement des maîtres, ces suggestions font état d'une conviction en la capacité des enseignants à faire de la recherche.

Ici au Québec, le seul essai d'intégration s'est limité à l'introduction de la *recherche-action* dans le programme PERMAMA (Perfectionnement des Maîtres en Mathématiques). Cela semble un excellent moyen de permettre à l'enseignant de prendre conscience de ses capacités à résoudre des problèmes pédagogiques, mais ce genre de recherche demeure vulnérable aux critiques visant des méthodologies jugées non-scientifiques (Commission d'études, p. 52).

En somme, en aucun pays a-t-on réussi à vraiment intégrer de façon systématique, rationnelle et scientifique la recherche à la formation et au perfectionnement des enseignants. Selon Bauersfeld (1976), *tant qu'on ne transformera pas les programmes de formation et de perfectionnement, la conception de l'enseignant-chercheur demeurera une utopie*. Néanmoins, avant de proposer des modifications de programmes faut-il identifier les éléments de l'activité de recherche les plus profitables aux enseignants.

4 ÉLÉMENTS DE LA RECHERCHE UTILISABLES PAR L'ENSEIGNANT.

Comme nous l'avons mentionné auparavant, les éléments de la recherche utiles à l'enseignant dans sa classe doivent être jugés en fonction des exigences de la discipline considérée. En mathématique, il ne suffit pas de savoir qu'un élève a trouvé la réponse à un problème car, au point de vue pédagogique, il est tout aussi important de connaître les processus de pensée employés. Pour y arriver, l'enseignant doit pouvoir analyser l'aspect cognitif de la matière, tant au point de vue théorique que pratique. Nous décrivons ci-dessous les éléments de la recherche susceptibles de l'aider à analyser une situation pédagogique à l'aide de modèles de compréhension et de modèles d'apprentissage (aspects théoriques), puis, à planifier ses interventions et à les réaliser à l'aide de l'entrevue clinique et de l'expérimentation didactique (aspects pratiques).

4.1 ASPECTS THÉORIQUES

L'apprentissage de la mathématique requiert bien plus que la simple mémorisation de règles. Ce *plus* peut être décrit comme signification, compréhension, *insight*, adaptation à la réalité, etc. Bauersfeld (1976) indique qu'il est important de distinguer entre la structure mathématique ("la matière signifiée"), la transformation de celle-ci par l'enseignant ("la matière enseignée"), et son assimilation par l'élève ("la matière apprise").

Pour que l'enseignant puisse prendre conscience de ces distinctions, il doit posséder les moyens de faire l'analyse conceptuelle d'une notion, et pour qu'une telle analyse ait une portée didactique, elle doit se situer dans un contexte cognitif et pédagogique. Il est donc essentiel, dans un premier temps, de lui présenter des modèles théoriques qui lui permettent de décrire les composantes de la compréhension et de l'apprentissage.

De tels modèles ont été spécialement conçus pour décrire différents modes de compréhension et pour identifier les différentes étapes de l'apprentissage des concepts mathématiques. Evidemment ce n'est qu'en appliquant ces modèles à l'analyse de notions précises telles que l'aire, ou la numération, que l'enseignant pourra se décentrer des produits de l'apprentissage (réponses écrites) pour s'attacher davantage aux processus de pensée.

L'exemple suivant, une analyse de la notion d'aire au moyen des modèles de la compréhension, permettra de mieux apprécier la puissance de cet outil didactique.

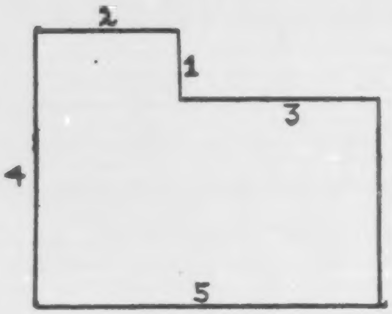
Modes de compréhension de la notion d'aire

Bruner (1960) fut le premier à décrire deux modes de compréhension en contrastant la **pensée analytique** (étapes explicites, pleine conscience des opérations et des informations pertinentes) à la **pensée intuitive** (perception globale et implicite d'un problème, inconscience des processus utilisés pour l'obtention d'une bonne ou d'une mauvaise réponse). Skemp (1976) a fait la distinction entre la compréhension **instrumentale** (l'application de règles sans raisons) et la compréhension **relationnelle** (savoir quoi faire et pourquoi). Jugeant ces deux modèles complémentaires, Byers et Herscovics (1977) les ont réunis en un nouveau modèle qui incorpore la notion de compréhension analytique à celle de compréhension **formelle** caractérisée

- par la capacité à relier les symboles et la notation aux idées mathématiques pertinentes;
- par la capacité à combiner les idées mathématiques dans un enchaînement de raisonnements logiques.

Ces modèles de la compréhension sont accessibles et utiles aux enseignants. Cela nous a été démontré lors d'un atelier réunissant un groupe d'enseignants-conseillers pédagogiques du primaire (Herscovics, 1978). En moins de deux heures ils ont pu identifier divers modes de compréhension de la notion d'aire à partir d'une discussion de l'erreur suivante.

La plupart des enfants vont donner "20" comme aire d'un rectangle de dimension 4 par 5. Mais peut-on inférer qu'ils comprennent quand on sait que, même à la fin du primaire, un grand nombre d'élèves donneront la même réponse pour l'aire de la figure ci-dessous?



Ces enfants font preuve d'une compréhension instrumentale de la notion d'aire du rectangle qu'ils généralisent incorrectement à d'autres figures. Le rectangle étant une figure trop spécialisée pour leur permettre de construire la notion générale de l'aire, des figures arbitraires, telles la feuille d'érable, doivent être utilisées dès le début (Bergeron J.C., Green A., 1969).

Bien que facile, la notion générale de l'aire est souvent confondue avec la notion de surface: on peut toucher à la surface mais on ne peut pas toucher à son aire qui est la **mesure de la surface**. Une compréhension **intuitive** de ce concept se fonde sur l'idée de **recouvrement** en posant la question de **quantité**, mais sans entrer dans celle de mesure. Plusieurs moyens peuvent amener l'enfant à en prendre conscience: activités de coloriage de figures, de découpage de tissus, de recouvrement de livres, etc. Evidemment, nous écartons les recouvrements partiels (i.e. par des pièces de monnaie) et les recouvrements superposés. Des problèmes peuvent être posés sur l'invariance de l'aire, sujette à certaines transformations de la figure donnée, i.e. faudra-t-il plus de peinture pour colorier une figure lorsqu'elle est découpée en morceaux?

Une compréhension **relationnelle** se manifesterait par une quantification mesurable de la notion de recouvrement. Une figure comme la feuille d'érable serait recouverte de carrés qu'il s'agirait de compter. Comme le recouvrement ne peut se faire exactement, la réponse trouvée ne serait qu'approximative, quitte à être précisée par l'emploi d'unités de mesure de plus en plus petites.

Cette technique de recouvrement suivie d'un comptage peut aussi s'apprendre d'une façon instrumentale. Ceci peut être vérifié indirectement en donnant à l'enfant moins de carrés que nécessaires pour recouvrir la figure donnée. Si celui-ci donne comme mesure de l'aire le nombre de carrés en sa possession (sans que ceux-ci couvrent complètement la surface), il y aurait évidence d'une compréhension **instrumentale** (compter des carrés).

Une compréhension **formelle** de la notion générale de l'aire serait celle représentée par l'intégrale de Riumann ($\int f(x)dx.$) (évidemment, il ne s'agit pas au primaire d'introduire la notion de la limite d'une suite de sommes de Riemann).

Étapes d'apprentissage de la notion d'aire

Vu l'échec en didactique de la mathématique des modèles d'apprentissage basés sur une théorie behavioriste, notre approche se place carrément dans le contexte de la psychologie cognitive.

Il n'y a pas de modèle d'apprentissage universel s'appliquant à tous les niveaux scolaires et à toutes les activités mathématiques. La théorie du développement intellectuel proposée par Bruner (1966, 1973) semble particulièrement utile pour décrire la formation de concepts au primaire; le modèle d'apprentissage de Dienes (1970) s'applique surtout à l'enseignement de structures algébriques; celui de Herscovics (1979) n'a été utilisé jusqu'à présent que pour l'apprentissage de l'algèbre au secondaire.

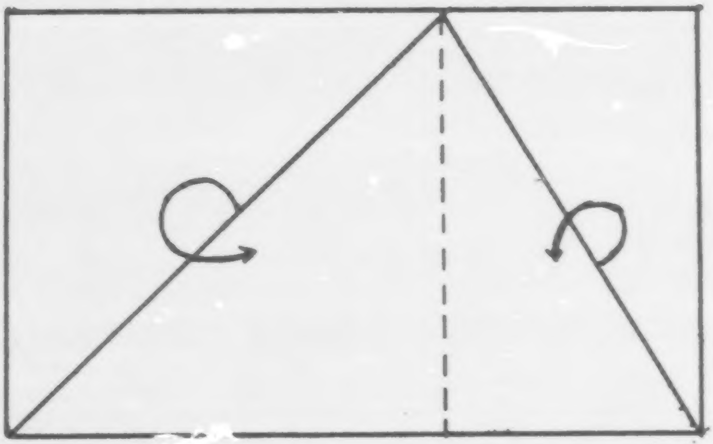
Comme un modèle cognitif de l'apprentissage ne peut ignorer les modes de compréhension, de même il ne peut ignorer les modes de représentation. Bruner en conçoit trois: le mode de représentation par l'action (*enactive*), le mode de représentation par l'image (*iconic*), et le mode de représentation par le symbole, les symboles n'étant plus une image de l'objet (*symbolic*). Chacun de ces moyens de représentation peut condenser différentes quantités de connaissances mais leur emploi dépend de l'âge ainsi que du développement intellectuel de l'enfant qui, normalement, passerait par ces trois modes de représentation, dans exactement cet ordre. Il nous met en garde contre un enseignement trop hâtivement centré sur le symbolisme. D'après lui, un enfant qui n'a pas vécu assez d'expériences avec les deux autres modes n'a aucun moyen de s'en sortir lorsque ses opérations symboliques ne lui permettent pas de résoudre son problème. La notion d'aire, discutée plus haut, servira à préciser les étapes correspondant à notre modèle d'apprentissage qui est basé sur ces trois modes de représentation.

Les activités de recouvrement et de découpage se situent sans contredit au niveau de la représentation par l'action. Même lorsqu'il s'agit de mesurer, il faudrait que, dans un premier temps, l'enfant ait l'occasion de recouvrir plusieurs figures arbitraires avec des petits carrés qu'il peut compter. L'aire d'une surface, sujette à certains changements produits par découpage, par rotation, par translation, etc., deviendra ainsi quantifiable, et elle apparaîtra comme invariante sous ces transformations, renforçant par le fait même l'assimilation de cette notion. Ce n'est qu'après beaucoup d'activités concrètes et variées de recouvrement et de mesure de l'aire correspondante que la représentation iconique serait utilisée.

L'enfant pourrait maintenant recouvrir la figure étudiée d'un quadrillage (images des carrés) et compter tout simplement les carrés du recouvrement. L'emploi d'un quadrillage de plus en plus fin améliorera l'approximation. C'est là l'essence de la notion de l'intégrale de Riemann, laquelle n'est abordée qu'au niveau collégial. Mais c'est avec des figures plus spécialisées que nous pouvons atteindre une certaine formalisation.

En effet, ayant acquis la notion générale de l'aire, l'enfant peut maintenant l'appliquer à un rectangle en comptant les carrés utilisés pour le recouvrir. Si ensuite on le fait immédiatement passer à la formule de l'aire, celle-ci n'aurait alors pour lui d'autre justification que de donner la même réponse sans qu'il puisse expliquer pourquoi (i.e. compréhension **instrumentale** de la formule). Une meilleure compréhension peut être obtenue lorsqu'on porte l'attention de l'élève sur le fait que chaque rangée comporte le même nombre de carrés. Donc la somme totale peut être obtenue par l'addition répétée qui se traduirait par une multiplication du nombre de rangées par le nombre de colonnes. Ceci justifierait la validité de la formule **Aire = base x hauteur** d'où une compréhension **formelle** de l'aire du rectangle.

L'acquisition d'une compréhension formelle de l'aire du rectangle peut à son tour servir de base pour justifier la formule de l'aire d'autres figures géométriques telles le triangle, le parallélogramme, le losange, etc.. Par exemple, il est facile de faire construire par l'enfant un rectangle ayant la même base et la même hauteur qu'un triangle donné. Il ne s'agit alors que de découper les morceaux du rectangle et de les superposer sur le triangle pour découvrir que l'aire du triangle est exactement la moitié de l'aire du rectangle, soit **Aire=1/2 (base x hauteur)**



4.2 ASPECTS PRATIQUES

Les modèles de la compréhension permettent à l'enseignant de s'orienter vers l'aspect cognitif de l'apprentissage. Cependant, la validité de ses analyses demeurera toujours hypothétique tant qu'il ne les aura pas vérifiées dans la pratique en les confrontant à la démarche intellectuelle de l'élève. Evidemment, une recherche qui vise à pénétrer la pensée de l'étudiant implique nécessairement l'étude de cas. Les techniques courantes employées dans de telles études sont l'entrevue clinique, qui permet de faire un diagnostic (Opper, 1977; Easley,1977) et l'expérimentation didactique soviétique (Menchinskaya, 1969; Kantowski, 1979), qui incorpore la dimension *enseignement* à l'entrevue clinique, permettant ainsi d'étudier sur le vif pendant qu'ils se forment, les processus de pensée de l'élève.

L'entrevue clinique

On sait que la plupart des erreurs que font les enfants en mathématique ne sont pas dues à l'inattention, mais bien à de fausses règles qu'ils se sont construites, règles qui leur paraissent parfaitement logiques (Erlwanger, 1975; Ginsburg, 1977). De même, il est bien connu qu'à la base des difficultés qu'ils éprouvent à développer certaines habiletés mathématiques, on retrouve des difficultés à construire les schèmes qui les soustendent. Dans le premier cas, les élèves peuvent nous indiquer les règles qu'ils appliquent, mais dans le second, on ne peut pas s'attendre à ce qu'ils nous communiquent les schèmes qu'ils n'ont pas encore construits.

Par contre, pour l'aider à poursuivre la construction des schèmes amorcés et à corriger ses fausses règles, l'enseignant a besoin de déterminer le niveau du développement cognitif de l'enfant. Nous croyons que ce n'est qu'à travers un questionnement flexible et subtil, *l'entrevue clinique*, que le maître peut parvenir à cerner ce développement cognitif et amener l'enfant à dévoiler les causes de ses difficultés.

L'entrevue clinique se prête particulièrement bien à l'étude des processus cognitifs (Easley, 1977). Cette technique de recherche, mise au point par Piaget, est spécialement conçue pour mettre à jour des témoignages sur les opérations intellectuelles en jeu (Opper, 1977). L'expérimentateur qui dirige l'entrevue peut, suivant un questionnement standardisé ou semi-standardisé, interroger le sujet sur ses actions, ses démarches, ses hypothèses, la généralité ou la réversibilité de ses concepts.

Cette technique étant bien connue de tous les chercheurs, nous serions mal venus de consacrer plus de temps à la décrire. Toutefois, l'exemple suivant en confirmera l'intérêt pour l'enseignant.

Le graphe d'une équation à deux variables est défini comme étant l'ensemble (infini) de tous les points du plan dont les coordonnées correspondent aux solutions de l'équation. Ainsi, le graphe d'une équation linéaire consiste en un ensemble de points formant une droite. Ceci ne pose aucun problème aux enseignants qui eux, conçoivent la droite comme un ensemble de points, mais tel n'est pas le cas pour la majorité des élèves (Kerslake, 1977). En effet, si on demande aux étudiants combien de points contient un segment de droite donné, ils peuvent tout aussi bien répondre *aucun*, *un*, *deux*, ou *trente*! Des entrevues cliniques de quelques élèves du niveau du Secondaire III ont montré que ces réponses, loin d'être idiotes, reflétaient leur perception (Herscovics, 1979): le sujet répondant *aucun* ne voyait nullement le rapport entre le point et la droite; celui répondant *un* expliquait qu'il pensait au point milieu; et pour celui répondant *deux* il s'agissait des points terminaux du segment.

A l'occasion de ces travaux nous avons repris une expérience de Piaget et Inhelder (1947) sur les notions de point et de continuité. La tâche consistait à subdiviser un segment de droite, puis à répéter l'opération indéfiniment dans le but d'imaginer la forme vers laquelle tendrait le morceau ultime. Voici un extrait de l'entrevue d'un sujet après qu'il eut fait quelques subdivisions:

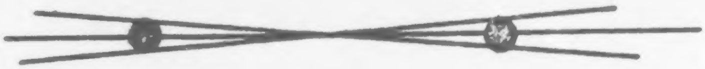
Q. Peux-tu continuer à subdiviser encore longtemps?
R. **Hum, non, pas trop longtemps.**
Q. Combien de fois?
R. **Environ 4 ou 5 fois.**
Q. Pourquoi?
R. **Ca deviendra trop petit pour travailler avec.**
Q. Et si tu imagines que c'est un élastique et que tu peux l'étirer?
R. **Si on l'étire jusqu'à ce que c'est aussi long qu'un des morceaux on pourra le faire (diviser) autant de fois qu'avant.**
Q. Et qu'est-ce qui restera à la fin?
R. **Quelque chose de si petit qu'on ne pourra pas le voir.**
Q. Et dans ce petit bout, il y a combien de points?
R. (surpris) **Des points?**
Q. Penses-tu que le dernier morceau aura une forme?
R. **Oui, si on le regarde au microscope.**
Q. Et quelle forme penses-tu qu'il aura?
R. **Un carré.**
Q. Un carré?
R. **Je pense que oui, car une ligne a une épaisseur.**

Revenant alors à la notion de point:

Q. Voici deux points. Combien de points penses-tu pouvoir mettre entre ces deux-ci?
R. **Même grandeur... environ 30.**
Q. Et s'ils étaient plus petits?
R. **S'ils étaient plus petits de moitié...60.**
Q. Et s'ils étaient encore plus petits?
R. **Vous pouvez, j'imagine, vous pouvez y mettre des milliers.**
Q. Et ça donnerait?
R. **Ca ressemblerait à une droite, une droite.**
Q. Tu me dis que ces points te donnent une droite, mais qu'une droite n'est pas faite de points!
R. **C'est vraiment difficile de dire ce que c'est un point. Car il y a des gens qui tracent un point comme ça** (fait un point avec son crayon). **Un point veut dire le centre, comme ça. Donc je pense qu'une "droite de points" ne serait pas plat comme ceci** (indique une droite tracée à la règle). **Mais si vous avez**

un point et un autre point, c'est comme une vague (il dessine).

Il est évident que cet élève n'avait pas fait abstraction de la notion de point et qu'il la confondait avec celle de disque, c'est à dire "un point ayant une dimension". Une telle confusion pouvait faire dire au sujet mentionné ci-dessus *que par deux points on peut faire passer plusieurs droites* , ce qu'il expliquait par le dessin suivant.



Expérimentation didactique

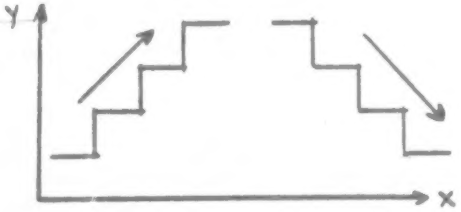
Bien que l'entrevue clinique se révèle un bon instrument diagnostique de la pensée de l'élève, elle ne fait pas intervenir la composante *enseignement* qui, pour nous, est un moyen d'aider l'élève à construire ses schèmes. L'enseignant qui prépare sa leçon présume qu'elle peut être suivie pas à pas par celui-ci. Mais, le grand nombre d'enfants manifestant des difficultés d'apprentissage indique que ce n'est pas toujours le cas. Une façon de remédier à ce problème serait de vérifier à chaque pas, par un questionnement approprié, l'opportunité de modifier son plan d'instruction.

La méthode dite *expérimentation didactique soviétique* incorpore à l'entrevue clinique la dimension *enseignement* et permet de suivre les schèmes de l'enfant pendant qu'ils se forment. Cette méthode porte sur de petits échantillons d'élèves soumis à un enseignement. Les données sont obtenues dans un cadre clinique par enregistrement des réponses écrites et verbales, puis analysées selon des critères qualitatifs. Elle est longitudinale et elle implique plusieurs rencontres pendant lesquelles le plan d'enseignement est constamment modifié suivant les observations recueillies (Kantowski, 1979).

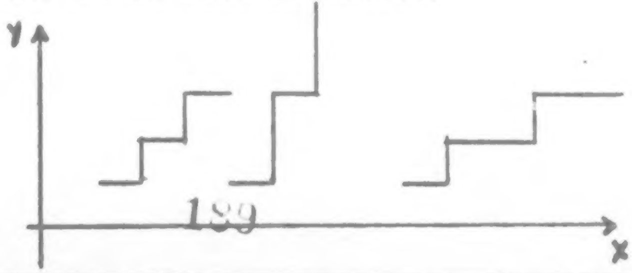
Cette technique, mise au point par des chercheurs soviétiques, se centre sur les aspects qualitatifs de la pensée et de l'apprentissage. Elle vise à reproduire d'une façon systématique les processus de pensée et à les étudier sur le vif alors qu'ils se forment sous l'influence de divers enseignements. Cette méthode permet à la fois de découvrir les changements qui ont lieu au cours des leçons et de suivre chez un même élève l'évolution du processus mental (Menchinskaya, 1969 a,b).

Nous pouvons illustrer la valeur de l'expérimentation didactique en décrivant quatre problèmes pédagogiques dont nous avons pris conscience lors d'une expérience portant sur la construction de la notion de pente (Herscovics, 1980). Ces quatre problèmes peuvent paraître mineurs à l'enseignant, qui lui pense d'une façon formelle, mais chacun d'eux peut s'avérer un obstacle majeur pour l'élève.

1. Nous désirions introduire de façon concrète la notion de pente d'une droite en la reliant à la notion de pente d'un escalier. Nous avions demandé à trois étudiants du niveau de secondaire III de dessiner dans un plan cartésien un **escalier montant**. Il ne nous était pas venu à l'esprit qu'aucune confusion soit possible étant donné que le sens positif des axes détermine de façon univoque le sens d'un escalier montant. Ce n'est que lorsque les sujets nous ont signifié qu'ils pouvaient tout aussi bien descendre que monter un même escalier que nous nous sommes rendus compte du besoin d'expliciter la convention habituelle.

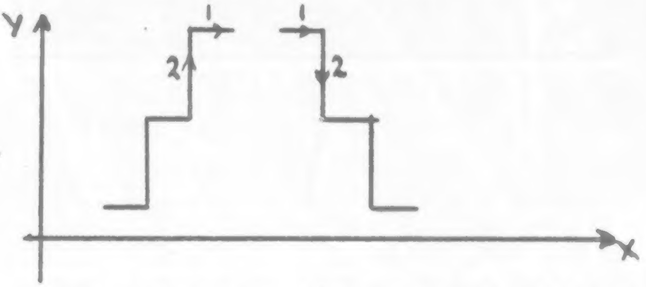


2. Nous avions demandé aux élèves de dessiner trois escaliers montants en variant la dimension du giron (partie horizontale de la marche) et celle de la contremarche (partie verticale). Nous voulions des escaliers dont la contremarche soit égale au giron, soit le double du giron, ou en soit la moitié.



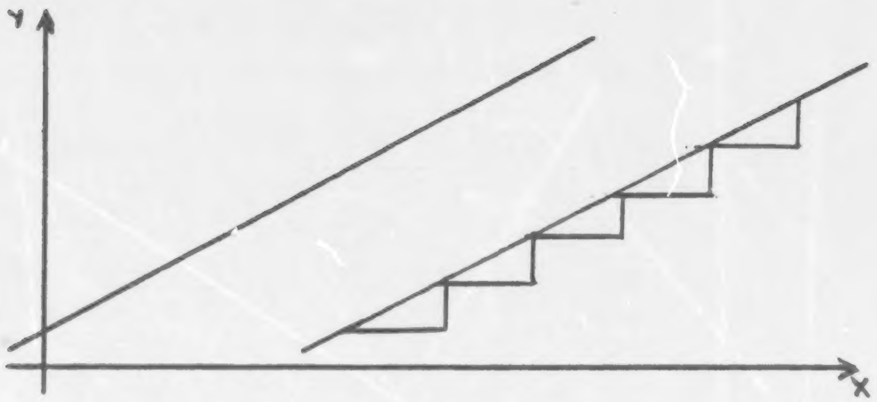
En les comparant, les étudiants nous ont dit que la pente d'un escalier dépendait à la fois de la longueur de la contremarche (c) et de celle du giron (g). En leur demandant ensuite d'exprimer cette dépendance nous nous attendions à ce qu'ils l'expriment sous forme d'un rapport ($\frac{c}{g}$ ou $\frac{g}{c}$). Cependant, c'est sous forme d'équation qu'ils se sont exprimés (i.e. **2g = c** dans le cas du deuxième escalier). Evidemment, le passage de l'équation au rapport $\frac{c}{g}$ qui définit la pente a pu se faire facilement.

3. Nos élèves avaient pris conscience du besoin d'une convention pour distinguer entre la pente d'un escalier montant et celle d'un escalier descendant et avaient accepté la convention habituelle (positive dans le premier cas, et négative dans le second).



Contrairement à notre attente, ils ont par la suite ignoré cette convention qu'il fallait sans cesse leur rappeler. Ce n'est qu'en inversant le problème (dessiner un escalier correspondant à une pente négative) que la convention a été maîtrisée.

4. En posant la question "Est-ce qu'il est sensé de parler de la pente d'une droite?" les élèves se sont bien rendus compte que cela était relié au travail précédent sur les escaliers. Mais aucun d'eux n'a pu faire le lien. Même la demande "d'attacher un escalier à une droite donnée" n'a rien produit et il a fallu de fait le dessiner nous-mêmes. La pente de la droite a pu être alors définie comme étant **la pente de l'escalier qui s'y rattache.**



Ces quatre exemples montrent que la nature formelle de la pensée de l'enseignant l'empêche souvent de prévoir si ses étudiants peuvent le suivre dans sa présentation. L'expérimentation didactique lui permet de prendre conscience des difficultés qu'ils éprouvent et d'ajuster en conséquence son intervention pédagogique.

5 CONCLUSIONS

Nous avons identifié et illustré par des exemples les éléments de la recherche que nous croyons utiles aux maîtres pour l'enseignement de la mathématique.Ainsi, avons-nous concrétisé les recommandations de la Commission d'Etude sur les Universités du Québec qui, comme nous, souhaite l'intégration de la recherche à la formation et au perfectionnement des enseignants dans le but d'améliorer la qualité de l'éducation.

Notre approche découle d'une philosophie qui conçoit l'enseignant comme le levier principal permettant de rehausser la valeur éducative des écoles.Cette conviction s'appuie sur l'expérience des vingt dernières années dans notre discipline alors que des millions ont été investis presqu'exclusivement dans l'élaboration de programmes et de textes scolaires, sans obtenir les résultats escomptés.Ces échecs peuvent être attribués à l'ignorance de la psycho-pédagogie de l'apprentissage et au manque de préparation de l'enseignant. Pourtant, il est bien évident qu'aucun programme ou texte scolaire ne peut en soi répondre adéquatement à tous les problèmes rencontrés en classe.Le besoin de former des enseignants autonomes et capables d'innover n'est donc pas fictif, mais bien réel.

Il ne suffit pas à l'enseignant de vouloir être autonome et de vouloir innover: il doit en avoir les moyens. Les éléments de la recherche que nous favorisons les lui fourniraient. S'il peut les assimiler , il atteindra un niveau plus professionnel, celui *d'enseignant-chercheur*. L'application minutieuse des méthodes que nous suggérons rendra sa recherche plus scientifique, ce qui aura l'effet de la valoriser et d'en faciliter la communication. Un tel enseignant, tout en étant plus ouvert aux innovations qu'on lui recommande, deviendra aussi plus critique quant à leur portée pédagogique qu'il pourra vérifier lui-même par l'étude de cas.

C'est en fait l'étude de cas qui se situe au coeur de notre formule d'intégration. En effet, nous pensons qu'une initiation à cette pratique est essentielle pour se sensibiliser aux processus de pensée propres à l'enfant et pour analyser la valeur pédagogique d'une intervention. Les modèles de compréhension et d'apprentissage fournissent un cadre de référence à ces analyses et permettent une évaluation plus fine dépassant le simple niveau des habiletés instrumentales. Une telle évaluation est indispensable à l'enseignant qui doit fonctionner dans une classe hétérogène et se préoccuper de la croissance optimale de chaque élève. La tendance actuelle à intégrer aux classes régulières les enfants en troubles d'apprentissage ne fait qu'augmenter les exigences auxquelles il doit répondre. En effet, selon la Direction Générale Supérieure du Ministère de l'Éducation du Québec (MEQ, Paré, 1979), il faudrait que dans le cadre du baccalauréat spécialisé en Éducation préscolaire et en Enseignement primaire, tous les candidats soient préparés à faire un travail de prévention, de dépistage et de correction des troubles mineurs d'adaptation et d'apprentissage.

Les éléments de la recherche auxquels nous nous sommes attachés débordent le domaine de la mathématique et ils pourraient être appliqués sans trop de modification aux programmes de formation et de perfectionnement en didactique des sciences de la nature, par exemple. En sciences humaines, par contre, il appert à prime abord qu'au moins l'entrevue clinique et l'expérimentation didactique pourraient s'avérer profitables mais c'est en fin de compte aux didacticiens de ces disciplines d'en juger.

Si dans un premier temps il s'agit de déterminer les aspects de la recherche utiles à la pratique de l'enseignement, il faut dans un deuxième temps étudier la question tout aussi importante touchant leur assimilation par les maîtres. Cette question a été sondée au cours de quelques études-pilotes. Dans certains de nos cours de formation et de perfectionnement, tant au niveau primaire que secondaire, la possibilité d'enseigner à nos étudiants les modèles de compréhension et d'apprentissage, et de les initier à l'entrevue clinique a été explorée. Des résultats assez prometteurs justifient une étude systématique de la question. De plus, une version de l'expérimentation didactique que nous jugeons à la portée des enseignants a été conçue (Bergeron,Herscovics, 1979).

Nous avons entrepris un projet de recherche qui s'échelonne sur trois ans et qui porte sur la vérification d'hypothèses concernant des moyens concrets d'intégrer ces éléments de la recherche à la formation et au perfectionnement des maîtres, et la façon d'évaluer les progrès accomplis.

Nous serions heureux d'échanger des commentaires et de recevoir de nos collègues, leurs critiques et suggestions.

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CMESG/GCEDM 1980
Special Group I

PROBLEM-SOLVING

E.J. Barbeau

In Grade 10, I had a course in North American geography. This was made particularly vivid by a teacher who had travelled extensively in Canada and the United States, and could give a first-hand account of the many regions. Likewise, I had a French teacher who had lived for a time in a French-speaking country. And again, there are English teachers who have acted in plays, music teachers who perform in groups and biology teachers who are active naturalists. How many mathematics teachers are there who have done some mathematical exploration of their own, and are thus able to bring to their classes the same immediacy brought by teachers of the other subjects described above?

The current emphasis on problem-solving worries me. Perhaps a third of the talks at a conference or papers in a journal are addressed to this topic. The yearbook of the NCTM focusses on this area, and the publishers are moving in with a vengeance. Problem-solving promises to be in the 80s what New Math was in the 60s. "Polya" will replace "Piaget" as the buzzword among educators. There is a danger that it will become a fad, a religion, with its stereotyped techniques and compelling prophets. Will it be reduced to a bag of jargon and procedures to be imposed upon our hapless students?

We are not wrong to exalt problem-solving in the curriculum. Like the New Math of the 60s, problem-solving promises to overcome sterility in much of our teaching bv catching students up in the process of doing mathematics and permitting them the satisfaction of understanding what is occurring. The abuses I have just outlined can be avoided if the teacher is zealous in mediating through his own personal experience and techniques and theories bombarding him/her from all directions. Before all, the teacher should allow time for solving problems on his/her own. These need not be advanced and should spin off lots of material to share with a class or club.

The following examples are suitable for teachers or for students in high schools.

Number theory

1. Egyptian fractions. (a) Express 2/n, where n is an odd integer, as the sum of distinct reciprocals of positive integers. Representations which are short and involve smaller denominators are preferred.

   (b) For which values of n can 3/n be written as the sum of two distinct reciprocals of positive integers?

   (c) Paul Erdos has conjectured that for each positive integer n, except 1 and 2, 4/n is equal to the sum of three distinct reciprocals of positive integers. How far can you go towards establishing this?

   (Children may use different stategies to get the required representations, and should be encouraged to describe them.)

2. Summing powers of digits. (a) Take any number. Sum the digits. Repeat the process. What eventually happens? Here is a problem from an International Olympiad paper: Start with 4444^{4444}. Sum its digits to get A. (Approximately how many digits does 4444^{4444} have? Roughly, how big is A?) Sum the digits of A to get B. (How big is B?). Sum the digits of B to get C. Show that C must be a single digit number. (You have to do two things: get an idea of the size of C, and find its remainder when you divide by 9. This remainder will be the same as that obtained when you divide 4444^{4444} by 9, because of "casting out 9s".)

   (b) Is any number equal to the sum of the squares of its digits? What happens if you start with any number and repeat the process of summing the squares of its digits over and over?

   (c) The same as (b) with the word "squares" replaced by "cubes".

3. Iterations. (a) Start with any positive integer and perform this operation: if it is even, divide by 2; if it is odd, multiply by 3 and add 1. What eventually happens if you repeat this operation over and over? (If we start with 7, the chain obtained is 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, ...

   (b) Write four numbers at the corner of a square. Form a new square array who members are the absolute values of differences of adjacent members of the original square. What eventually happens? How do you know? (The square array (31, 46, 12, 8) leads successively to (15, 34, 4, 23), (19, 30, 19, 8), (11, 11, 11, 11), (0, 0, 0, 0).)

(c) Take a 4-digit number. (Allow 0 as a first digit.) Reverse its digits and take the difference of the two numbers. Repeat. What eventually happens?

4. The sets (1, 2, 3) and (1, 2, 2, 4) have the interesting property that the sum of the cubes of the members of each set is equal to the square of the sum of the members. Can you find other such sets?
5. Multiply 142857 by 1, 2, 3, 4, 5, 6, 7 and see what happens. Can you account for the phenomenon observed? Can you find other numbers with similar behaviour?
6. Given two positive integers, the product of their greatest common divisor and least common multiple is equal to the product of the two numbers. How do you account for this?

Geometry

7. You have a square chocolate cake which is uniformly iced on top and down its sides. Show how to divide it among nine people so that each slice is connected and each person gets exactly the same amount of cake and icing.
8. Show how to cut a cube by a plane in order to get a cross-section which is a regular hexagon.
9. Show how to cut a regular tetrahedron with a single straight slice to get two congruent (identical) halves. Identify the cross-section figure.
10. What is the angle between two adjoining face diagonals of a cube?
11. An isosceles triangle has its equal sides both of length 1 metre. What is the largest possible area it could have?
12. Given a quadrilateral, form a new one whose vertices are the midpoints of the sides of the original one. Can you say anything special about the second quadrilateral? Repeat the process. What happens?
13. Find the smallest number k such that every obtuse-angled triangle can be cut into k acute-angled triangular pieces.

Combinatorics

14. A man gets on a streetcar at one terminal loop and rides to the other terminal loop. The whole journey takes an hour and he passes 12 streetcars going the other way. How many streetcars are there plying the route? A woman comes to a stop and just misses a streetcar. How long can she expect to wait for the next one?

15. In the solitaire game of Clock, what is the probability of getting all the cards turned over?
16. You are given five weights of distinct magnitudes and an equal arm balance. What is the smallest number of comparisons necessary to ensure that the weights are arranged in ascending order of magnitude?
17. The game of SIM is played as follows. Put six dots on a page. Player A joins two dots with a red line. Then player B joins two dots with a green line. They play alternately, A using red, B using green and no pair of dots being joined more than once. Thus there are at most 15 moves. The winner is the first person to complete a triangle in his own colour (i.e. arrange that there are three of the given dots, any pair of which are joined by an edge of his colour). Can the game end in a draw? Can either player assure a win?
18. Play this game on a 4x4 grid of squares. Player A uses red and player B uses green and they play alternately. Each colours an uncoloured square with his own colour, but is prohibited from colouring any square adjacent along a side to a square he has already coloured. Thus, there can be no two adjacent red squares or two adjacent green squares. The loser is the first person without a possible move. Can the game end in a draw? Can either player assure a win?
19. Four couples meet for dinner. There is some shaking of hands. No one shakes hands with his/her spouse, and no pair shake hands more than once. When all this is done, one of the men asks each of the others how often he/she shook hands and gets a different answer from each. How often did his wife shake hands? (I got this one from H. L. Ridge, with whom I shared a masters course for teachers. After some initial puzzlement, the group was encouraged to try to model the situation, a procedure leading to both fun and enlightenment.)

This is but a sampler. Card tricks and puzzles of the Instant Insanity or string-and-ring topological variety that you can pick up in shops are often very good for the sort of mathematical reasoning which can lead to sudden flashes of insight. Mathematical recreations *and essays*, by Ball and Coxeter, is an excellent source of topics. Teachers should also patronize at least one journal that purveys problems; good choices are *Scientific American* (which has a regular column by Martin Gardner), *Crux Mathematicorum* (editor: F. G. B. Maskell, Algonquin College, Ottawa), *Journal of recreational mathematics* and *Mathematics magazine*. I have a bibliography and a collection of about 200 problems which I can send to anyone on request. Write to me at B201, University College, University of Toronto, Toronto M5S 1A1.

CMESG/GCEDM 1980
Special Group II

USE OF GATTEGNO'S NUMBER ARRAY

by
Martin Hoffman
Arthur Powell

On Tuesday, 10 June, at the 1980 conference of CMESG/GCEDM, an "ad hoc" session was held to examine a technique and a rationale for teaching different bases of numeration via a number array developed by Caleb Gattegno[1]. This technique had been parenthetically mentioned by Gattegno during the Saturday discussion session held after his conference lecture/seminar on "nothings"[2]. This technique is illustrative of how an awareness of the infinitesimal quantities of energy, which Gattegno calls "nothings", expended in thought processes could be applied within a science of mathematics education.

Lack of sufficient time to adequately investigate questions and concerns that participants raised during the discussion session stimulated Martin Hoffman and Arthur Powell to organize an "ad hoc" session. In addition to the expressed purpose for Gattegno's injection of this technique into the discussion session, we have found it to be a pragmatic tool in certain teaching situations. Having both worked with Gattegno and employed this technique in classrooms and in teacher training workshops, it was an opportunity for sharing experiences and exchanging viewpoints on the topic.

Gattegno's example concerned the following array of numerals:

1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900

and the simple introduction of a vertical line between any column or after the column headed by the 9. This device offers learners an immediate entry into any base of numeration.

With the above array and the use of a pointer, any numeral from 1 to 999 can be generated. For example, the numeral 572 can be generated by pointing to the 500 on the bottom row, the 70 on the middle row, and the 2 on the top row. The numeral 815 is generated by successively pointing to 800, 10, and 5. Using this technique, it is clear how the ordered sequence of numerals from 1 to 999 can be generated. As you try it, you can become aware of when and why you move from one numeral to the next and from employing one row to two and then to three rows.

Another activity can be created by inserting a vertical line between two columns in the array of numerals and imposing the single rule that only those signs which appear to its left are to be used. If the vertical line is placed between columns headed by the 7 and 8, as illustrated below, then this new system is that which is commonly referred to as the octal system of numeration or Base VIII.

1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900

In this activity, the common system of numeration, Base X, is represented by the state when the vertical line has been placed immediately after the column headed by the 9. Bases beyond Base X can also be studied[3].

In this approach, for each base of numeration, Roman numeral subscripts are used with numerals to indicate the magnitude of the sign "10". Once, however, it has been established that the domain or "world" of conversation is a given base, then reference to that base need not be given. For example, in Base V, the sign 13_V can be written and read "thirteen". This avoids the difficulties learners experience when 13 is read as "one three base five". The approach allows learners to develop feel for and insight into various bases of numeration without specific reference to the common base. The distinction between the concept of magnitude and the label given to that magnitude, which is obscured by the conventional way of reading numerals in bases other than the common one, is made more accessible. It also becomes clear that in all systems of numeration the "base" is in fact "10".[4]

Additionally, in the "ad hoc" session, time was devoted to developing an algorithm for performing subtraction in Base V. The algorithm is based on an awareness of complementary numbers; for example, complements in 10 and complements in (10 - 1) = 4[*]. The following unordered pairs represent the complements in 4 and 10, respectively.

(0,4), (1,3), (2,2), (3,1), (4,0)

(0,10), (1,4), (2,3), (3,2), (4,1), (10,0)

With this understanding, subtractions like $\begin{array}{r}100\\-21\\\hline 24\end{array}$ and $\begin{array}{r}10000\\-3413\\\hline 1032\end{array}$ can be done by inspection and without the need to "borrow".

One of the fundamental notions that mathematicians work with is that of transformations. For instance, in algebra and topology a consequence of this notion is isomorphism. An understanding of this applied to subraction in Base V can lead to the attitude, on the part of learners, that a given difficult problem can be transformed into an easier equivalent one. Thus, the transformations that one might perform on the problem

$\begin{array}{r}122\\-34\\\hline\end{array}$ can be to see it as $\begin{array}{r}123\\-40\\\hline\end{array}$ and then $\begin{array}{r}133\\-100\\\hline\end{array}$ so that the result 33 is immediately perceived.

The question that was then proposed was whether this approach could be applied in Base X, the common system of numeration. Since it is the mathematics or the 'algebra of the situation' which is invariant, only the arithmetic changes as one switches from one system of numeration to another. In the "ad hoc" session, the subtraction algorithm, based on complements and transformations, which had been developed in Base V was easily transferred to and applied in Base X.

It has been our experience that this approach and this way of naming numerals in different systems is more accessible to learners and teachers. Historically,

> "the incorporation of the study of bases of numeration in the elementary mathematics curriculum has led to a great deal of confusion for both teachers and learners. As usually presented, the topic is intro-

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[*] The domain of conversation, unless explicitly changed, is Base V.

> duced in a most illogical manner (ex: "base five is the base where there is no five"!) and the fundamental, and educationally valid, reasons for studying different bases of numeration are either neglected or misunderstood. This is evidenced by the common emphasis on exercises involving conversions from a given base to "base 10", which not only places unwarranted stress on this base, but also results in the side-stepping of valuable understandings into the world of numbers the study of bases can yeild. These exercises thus become one more instance of sterile, mechanical computations to be performed by students without insight into why they are being asked to undertake them."[5]

There are a number of situations in which we have discovered this approach to be an effective pedagogical tool. In our work with older students in programs designed for underprepared college students, remedial classes in elementary and secondary schools, and adults in continuing education programs it is an opportunity to re-examine areas which previouly presented difficulties. The study of fractions, decimals and operations on these objects, and quick calculations can be worked on in a base of numeration other than the common one. It presents a challenge with a novelty that they accept as worthwhile, and it avoids those reactions traditionally expressed when they are asked to "relearn" old material. In this way, "... teachers will enable them to overcome their feelings of frustration and boredom, while ensuring that they come in contact with the essential insight they need to make sense of operations and their properties."[6]

In teacher training workshops, in addition to the investigation of the consequences of a "world" in Base Z and becoming aware of the invariant algebra when applied to the common base of numeration, teachers are in a situation to examine the conventions of traditional mathematics curricula. Their stress on presenting mathematics as factual units, rather that as an intellectual activity in which perception and process are its foundations, can be realized to be that which undermines the enthusiasm that learners can have for mathematics.

References and Notes

1. Gattegno, Caleb. "Functioning as a Mathematician". Mathematics Teaching, 39, 1967.

2. ______________. "Among Mathematicians and Math Education Professors at Laval University". Educational Solutions 'Newsletter', Volume X, Number 1. 1980.

3. A complete discussion of the technique and its use for teaching numeration in any base and complementary addition and subtraction is contained in Gattegno's, The Common Sense of Teaching Mathematics. New York: Educational Solutions, Inc. 1974. pp 3 - 31.

4. Ibid; p.21.

5. Gattegno, Guy C. & Hoffman, Martin R. Handbook of Activities for the Teaching of Mathematics at the Elementary School. New York: Human Education, Inc. 1976. Section II-A6, p.1.

6. Ibid; Section II-A6, p.7.

MATHEMATICAL FILMS

Twice during the CMESG/GCEDM 1980 meeting, participants had an opportunity to review eight geometry films[1] produced by Caleb Gattegno. These sessions were not activities which were part of the deliberations of Working Group 3. Nevertheless, since they treated geometrical ideas, made an exciting appeal to ones sense of aesthetics, and stimulated spirited discussions of their potential implementation as pedagogical tools, it seems that a few brief comments are in order.

The Swiss mathematics educator, J. L. Nicolet, initiated the production and use of silent hand-animated geometrical drawings on cellulose acetate as an aid in his instruction to students 10 to 18 years old. He produced about thirty films of this type, each film lasting between 1 to 6 minutes. Nicolet's original methodology for teaching through his films was to have learners summarize that which was invariant in the dynamic images which passed before them on a screen. Thus, learners would produce for themselver that which the film maker had in mind.

Caleg Gattegno has synthesized Nicolet's original films, expanded their content, and recently employed computer-generated images for the animation of mathematical situations. His work has led to the production of seven new films under the general heading "Animated Geometry". In accordance with Nicolet's approach, these films are also silent and contain neither captions nor labels. Now, however, no particular methodology is suggested by Gattegno.

Following the viewings, the films were discussed with respect to their pedagogical utility and use in teacher training programs. They were seen as tools in the education of learners geometrical intuition, allowing for the generation of knowledge of spatial relationships through one's perception. Teachers could then seize the given opportunity to challenge learners to enter into a dialogue with oneself as to the situations being perceived in a film. In the process of such a dialogue, the unfolding of the story of the film occurs inside the learners. Thus, the wealth of dynamic geometrical images becomes a part of the experience and referential baggage of the learners.

The role of the teacher will naturally depend upon many variables: the size of the class, the age of the learners, the availability of paper, pencils, chalk, yarn, compasses, and the like. The films, being short, can be shown as many times as is required and time permits. Exercises, basing actions upon perception, will be suggested by the acquired awarenesses of the class.

These films are aids in furthering the goal of geometry teaching/learning as an activity of geometers, supporting a pedagogy that views geometry as a process rather than a product.

[1] The titles and showing times of the films viewed are:

1.	Families of Circles in the Plane	4 1/2 mins.
2.	Angles at the Circumference	6 1/2 mins.
3.	Common Definition of the Conics	4 mins.
4.	Locus of Points from which Two Circles are seen under the Same Angle	4 1/2 mins.
5.	Poles and Polars in the Circle	4 1/2 mins.
6.	Definitions of the Right Strophoid	4 mins.
7.	Epi- and Hypocycloids	4 mins.
8.	Foundations of Geometry	16 1/2 mins.

(Examples of the use of films with elementary school students and a discussion of geometrical intuition and mathematical films can be found in Volume II of Gattegno's For the Teaching of Mathematics.)

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